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AN ELEMENTARY TREATISE

ON

SOLID GEOMETRY

BY

CHARLES SMITH, M.A.
FELLOW AND TUTOR OF SIDNEY SUSSEX COLLEGE, CAMBRIDGE.

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PREFACE.

THE following work is intended as an introductory text-book on Solid Geometry, and I have endeavoured to present the elementary parts of the subject in as simple a manner as possible. Those who desire fuller information are referred to the more complete treatises of Dr Salmon and Dr Frost, to both of which I am largely indebted.

I have discussed the different surfaces which can be represented by the general equation of the second degree at an earlier stage than is sometimes adopted. I think that this arrangement is for many reasons the most satisfactory, and I do not believe that beginners will find it difficult.

The examples have been principally taken from recent University and College Examination papers; I have also included many interesting theorems of M. Chasles.

I am indebted to several of my friends, particularly to Mr S. L. Loney, B.A., and to Mr R. H. Piggott, B.A., Scholars of Sidney Sussex College, for their kindness in looking over the proof sheets, and for valuable suggestions.

CHARLES SMITH.

SIDNEY SUSSEX COLLEGE,
April, 1884.

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SOLID GEOMETRY.

CHAPTER I.

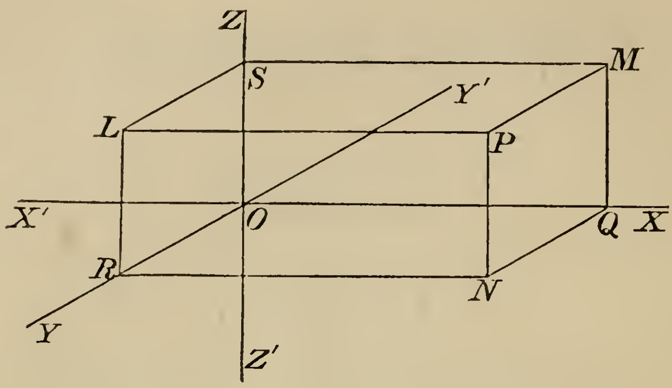
CO-ORDINATES.

1. The position of a point in space is usually determined by referring it to three fixed planes. The point of intersection of the planes is called the *origin*, the fixed planes are called the *co-ordinate planes*, and their lines of intersection the *co-ordinate axes*. The three *co-ordinates* of a point are its distances from each of the three co-ordinate planes, measured parallel to the lines of intersection of the other two. When the three co-ordinate planes, and therefore the three co-ordinate axes, are at right angles to each other, the axes are said to be *rectangular*.

2. The position of a point is completely determined when its co-ordinates are known. For, let YOZ, ZOX, XOY be the co-ordinate planes, and $X'OX$, $Y'OY$, $Z'OZ$ be the axes, and let LP, MP, NP, be the co-ordinates of P. The planes MPN, NPL, LPM are parallel respectively to YOZ, ZOX, XOY; if therefore they meet the axes in Q, R, S, as in the figure, we have a parallelopiped of which OP is a diagonal; and, since parallel edges of a parallelopiped are equal,

$$LP = OQ,\ MP = OR, \text{ and } NP = OS.$$

Hence, to find a point whose co-ordinates are given, we have only to take OQ, OR, OS equal to the given co-ordinates, and draw three planes through Q, R, S parallel respectively to the co-ordinate planes; then the point of intersection of these planes will be the point required.



If the co-ordinates of P parallel to OX, OY, OZ respectively be a, b, c, then P is said to be the point (a, b, c).

3. To determine the position of any point P it is not sufficient merely to know the absolute lengths of the lines LP, MP, NP, we must also know the directions in which they are drawn. If lines drawn in one direction be considered as positive, those drawn in the opposite direction must be considered as negative.

We shall consider that the directions OX, OY, OZ are positive.

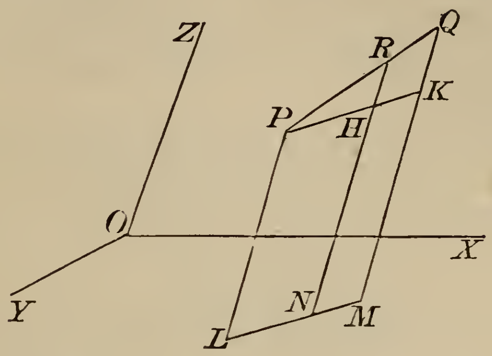
The whole of space is divided by the co-ordinate planes into eight compartments, namely $OXYZ$, $OX'YZ$, $OXY'Z$, $OXYZ'$, $OXY'Z'$, $OX'YZ'$, $OX'Y'Z$, and $OX'Y'Z'$.

If P be any point in the first compartment, there is a point in each of the other compartments whose absolute distances from the co-ordinate planes are equal to those of P; and, if P be (a, b, c) the other points are $(-a, b, c)$, $(a, -b, c)$, $(a, b, -c)$, $(a, -b, -c)$, $(-a, b, -c)$, $(-a, -b, c)$ and $(-a, -b, -c)$ respectively.

4. *To find the co-ordinates of the point which divides the straight line joining two given points in a given ratio.*

Let P, Q be the given points, and R the point which divides PQ in the given ratio $m_1 : m_2$.

Let P be (x_1, y_1, z_1), Q be (x_2, y_2, z_2), and R be (x, y, z).



Draw PL, QM, RN parallel to OZ meeting XOY in L, M, N. Then the points P, Q, R, L, M, N are clearly all in one plane, and a line through P parallel to LM will be in that plane, and will therefore meet QM, RN, in the points K, H suppose.

$$\text{Then } \frac{HR}{KQ} = \frac{PR}{PQ} = \frac{m_1}{m_1 + m_2}.$$

But $LP = z_1$, $MQ = z_2$, $NR = z$;

$$\therefore \qquad \frac{z - z_1}{z_2 - z_1} = \frac{m_1}{m_1 + m_2};$$

$$\therefore \qquad z = \frac{m_1 z_2 + m_2 z_1}{m_1 + m_2}.$$

Similarly
$$x = \frac{m_1 x_2 + m_2 x_1}{m_1 + m_2},$$

$$\text{and } y = \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}.$$

When PQ is divided *externally*, m_2 is negative.

The most useful case is where the line PQ is bisected: the co-ordinates of the point of bisection are

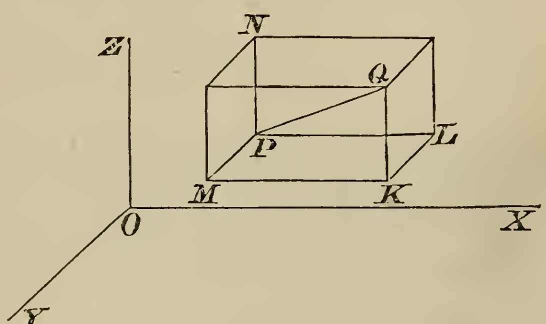
$$\tfrac{1}{2}(x_1+x_2),\ \tfrac{1}{2}(y_1+y_2),\ \tfrac{1}{2}(z_1+z_2).$$

The above results are true whatever the angles between the co-ordinate axes may be.

We shall in future consider the axes to be rectangular in all cases except when the contrary is expressly stated.

5. *To express the distance between two points in terms of their co-ordinates.*

Let P be the point (x_1, y_1, z_1) and Q the point (x_2, y_2, z_2). Draw through P and Q planes parallel to the co-ordinate planes, forming a parallelopiped whose diagonal is PQ.



Let the edges PL, LK, KQ be parallel respectively to OX, OY, OZ. Then since PL is perpendicular to the plane QKL, the angle PLQ is a right angle,

$$\begin{aligned}\therefore\ PQ^2 &= PL^2 + QL^2\\ &= PL^2 + LK^2 + KQ^2.\end{aligned}$$

Now PL is the difference of the distances of P and Q from the plane YOZ, so that we have $PL = x_2 - x_1$, and similarly for LK and KQ.

Hence $\quad PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \ldots\ldots$(i).

The distance of P from the origin can be obtained from the above by putting $x_2 = 0$, $y_2 = 0$, $z_2 = 0$. The result is

$$OP^2 = x_1^2 + y_1^2 + z_1^2 \ldots\ldots \text{(ii)}.$$

Ex. 1. The co-ordinates of the centre of gravity of the triangle whose angular points are (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) are $\frac{1}{3}(x_1+x_2+x_3)$, $\frac{1}{3}(y_1+y_2+y_3)$, and $\frac{1}{3}(z_1+z_2+z_3)$. √

Ex. 2. Shew that the three lines joining the middle points of opposite edges of a tetrahedron meet in a point. Shew also that this point is on the line joining any angular point to the centre of gravity of the opposite face, and divides that line in the ratio of 3 : 1. √

Ex. 3. Find the locus of points which are equidistant from the points (1, 2, 3) and (3, 2, −1). *Ans.* $x-2z=0$.

Ex. 4. Shew that the point $(\frac{3}{5}, 0, \frac{4}{5})$ is the centre of the sphere which passes through the four points (1, 2, 3), (3, 2, −1), (−1, 1, 2) and (1, −1, −2). √

6. Let α, β, γ be the angles which the line PQ makes with lines through P parallel to the axes of co-ordinates. Then, since in the figure to Art. 5 the angles PLQ, PMQ, PNQ are right angles, we have

$$PQ\cos\alpha = PL,$$
$$PQ\cos\beta = PM,$$
and
$$PQ\cos\gamma = PN.$$

Square and add, then

$$PQ^2\{\cos^2\alpha+\cos^2\beta+\cos^2\gamma\} = PL^2+PM^2+PN^2 = PQ^2.$$

Hence
$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma = 1.$$

The cosines of the angles which a straight line makes with the positive directions of the co-ordinate axes are called its *direction-cosines*, and we shall in future denote these cosines by the letters l, m, n.

From the above we see that any three direction-cosines are connected by the relation $l^2+m^2+n^2=1$. If the direction-cosines of PQ be l, m, n, it is easily seen that those of QP will be $-l$, $-m$, $-n$; and it is immaterial whether we consider l, m, n, or the same quantities with all the signs changed, as direction-cosines.

If we know that a, b, c are proportional to the direction-cosines of some line, we can at once find those direction-cosines. For we have $\frac{l}{a}=\frac{m}{b}=\frac{n}{c}$; hence each is equal to

$$\frac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{(a^2+b^2+c^2)}}, \text{ i.e. to } \frac{1}{\sqrt{(a^2+b^2+c^2)}}; \therefore l = \frac{a}{\sqrt{(a^2+b^2+c^2)}} \text{ \&c.}$$

Ex. The direction-cosines of a line are proportional to 3, -4, 12, find their actual values. *Ans.* $\frac{3}{13}, -\frac{4}{13}, \frac{12}{13}$.

7. The projection of a point on any line is the point where the line is met by a plane through the point perpendicular to the line. Thus, in the figure to Art. 2, Q, R, S are the projections of P on the lines OX, OY, OZ respectively.

The projection of a straight line of limited length on another straight line is the length intercepted between the projections of its extremities. If we have any number of points P, Q, R, S... whose projections on a straight line are p, q, r, s..., then the projections of PQ, QR, RS... on the line, are pq, qr, rs....

In estimating these projections we must consider the same direction as positive throughout, so that we shall always have $pq + qr + rs = ps$, that is the projection of PS on any line is equal to the algebraic sum of the projections of PQ, QR and RS. This result may be stated in a more general form as follows:—The algebraic sum of the projections of any number of sides of a polygon beginning at P and ending at Q is equal to the projection of PQ.

8. If we have any number of parallel straight lines, the projections of any other line PQ on them are the intercepts between planes through P and Q perpendicular to their directions. These intercepts are clearly all equal; hence the projections of any line on a series of parallel straight lines are all equal. And, since the projection of a straight line on an intersecting straight line is found by multiplying its length by the cosine of the angle between the lines, we have the following proposition:—

The projection of a finite straight line on any other straight line is equal to its length multiplied by the cosine of the angle between the lines.

9. In the figure to Art. 2, let $OQ = a$, $OR = b$, $OS = c$. Then it is clear that $x = a$ for all points on the plane $PMQN$, and that $y = b$ for all points on the plane $PNRL$,

and that $z = c$ for all points on the plane $PLSM$. Also along the line NP we have $x = a$, and $y = b$; and at the point P we have the three relations $x = a$, $y = b$, $z = c$.

So that a plane is determined by one equation, a straight line by two equations, and a point by three equations.

In general, any single equation of the form $F(x, y, z) = 0$, in which the variables are the co-ordinates of a point, represents a surface of some kind; two equations represent a curve, and three equations represent one or more points. This we proceed to prove.

10. Let two of the variables be absent, so that the equation of the surface is of the form $F(x) = 0$. Then the equation is equivalent to $(x - a)(x - b)(x - c) \ldots\ldots = 0$, where $a, b, c, \ldots$ are the roots of $F(x) = 0$; hence all the points whose co-ordinates satisfy the equation $F(x) = 0$ are on one or other of the *planes* $x - a = 0$, $x - b = 0$, $x - c = 0, \ldots\ldots$

Let one of the variables be absent, so that the equation is of the form $F(x, y) = 0$. Let P be any point in the plane $z = 0$ whose co-ordinates satisfy the equation $F(x, y) = 0$; then the co-ordinates of all points in the line through P parallel to the axis of z, are the same as those of P, so far as x and y are concerned; it therefore follows that all such points are on the surface. Hence the surface represented by the equation $F(x, y) = 0$ is traced out by a line which is always parallel to the axis of z, and which moves along the curve in the plane $z = 0$ defined by the equation $F(x, y) = 0$. Such a surface is called a cylindrical surface, or cylinder.

Next let the equation of the surface be $F(x, y, z) = 0$.

We have seen that all points for which $x = a$, and $y = b$ lie on a straight line parallel to the axis of z. Hence, if in the equation $F(x, y, z) = 0$, we put $x = a$, and $y = b$, the roots of the resulting equation in z will give the points in which the locus is met by a line through $(a, b, 0)$ parallel to the axis of z.

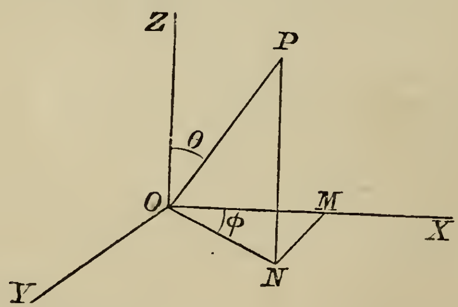
Since the number of roots is finite, the straight line will meet the locus in a finite number of points, and therefore the locus, which is the assemblage of all such points for different values of a and b, must be a surface and not a solid figure.

11. The points whose co-ordinates satisfy two equations must be on both the surfaces which those equations represent and therefore the locus is the *curve* determined by the intersection of the two surfaces. When three equations are given, we have sufficient equations to find the co-ordinates, although there may be more than one set of values, so that three equations represent one or more points.

12. The position of a point in space can be defined by other methods besides the one described in Art. 1.

Another method is the following: an origin O is taken, a fixed line OZ through O, and a fixed plane XOZ. The position of a point P is completely determined when its distance from the fixed point O, the angle ZOP, and the angle between the planes XOZ, and POZ are given. These co-ordinates are called Polar Co-ordinates, and are usually denoted by the symbols r, θ and ϕ, and the point is called the point (r, θ, ϕ).

If OX be perpendicular to OZ, and OY be perpendicular to the plane ZOX, we can express the rectangular co-ordinates of P in terms of its polar co-ordinates.



Draw PN perpendicular to the plane XOY, and NM perpendicular to OX, and join ON. Then

$$x = OM = ON \cos\phi = OP \sin\theta \cos\phi = r \sin\theta \cos\phi,$$
$$y = MN = ON \sin\phi = OP \sin\theta \sin\phi = r \sin\theta \sin\phi,$$
and $z = NP = OP \cos\theta = r \cos\theta$.

We can also express the polar co-ordinates of any point in terms of the rectangular. The values are,

$$r = \sqrt{(x^2 + y^2 + z^2)},\ \theta = \tan^{-1}\frac{\sqrt{(x^2 + y^2)}}{z},\ \text{and}\ \phi = \tan^{-1}\frac{y}{x}.$$

CHAPTER II.

THE PLANE.

13. *To shew that the surface represented by the general equation of the first degree is a plane.*

The most general equation of the first degree is

$$Ax + By + Cz + D = 0.$$

If (x_1, y_1, z_1) and (x_2, y_2, z_2) be any two points on the locus, we have

$$Ax_1 + By_1 + Cz_1 + D = 0,$$

and
$$Ax_2 + By_2 + Cz_2 + D = 0.$$

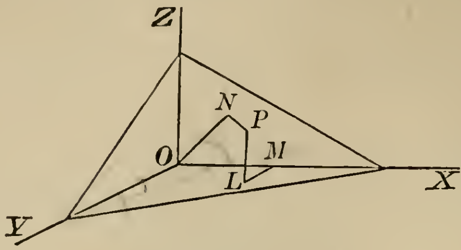
Multiply these in order by $\frac{m_2}{m_1 + m_2}$, and $\frac{m_1}{m_1 + m_2}$ and add; then we have

$$A\frac{m_2 x_1 + m_1 x_2}{m_1 + m_2} + B\frac{m_2 y_1 + m_1 y_2}{m_1 + m_2} + C\frac{m_2 z_1 + m_1 z_2}{m_1 + m_2} + D = 0.$$

This shews [Art. 4] that if the points (x_1, y_1, z_1), (x_2, y_2, z_2) be on the locus, any other point in the line joining them is also on the locus; this shews that the locus satisfies Euclid's definition of a *plane.*

14. *To find the equation of a plane.*

Let p be the length of the perpendicular ON from the origin on the plane, and let l, m, n be the direction-cosines of the perpendicular. Let P be any point on the plane, and draw PL perpendicular on XOY, and LM perpendicular to OX.



Then the projection of OP on ON is equal to the sum of the projections of OM, ML and LP on ON.

Hence if P be (x, y, z), we have

$$lx + my + nz = p \quad\text{.....................(i)},$$

the required equation.

By comparing the general equation of the first degree with (i), we see that the direction-cosines of the normal to the plane given by the general equation of the first degree are proportional to A, B, C; and therefore [Art. 6] are equal to

$$\frac{A}{\sqrt{(A^2+B^2+C^2)}}, \quad \frac{B}{\sqrt{(A^2+B^2+C^2)}}, \quad \frac{C}{\sqrt{(A^2+B^2+C^2)}}.$$

Also the perpendicular from the origin on the plane is equal to

$$\frac{-D}{\sqrt{(A^2+B^2+C^2)}}.$$

15. To find where the plane whose equation is

$$Ax + By + Cz + D = 0,$$

meets the axis of x we must put $y = z = 0$; hence if the intercept on the axis of x be a, we have $Aa + D = 0$.

Similarly if the intercepts on the other axes are b and c we have $Bb + D = 0$, and $Cc + D = 0$. Hence the equation of the plane is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

This equation can easily be obtained independently.

16. *To find the equation of the plane through three given points.*

Let the three points be (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3).

The general equation of a plane is

$$Ax + By + Cz + D = 0.$$

If the three given points are on this plane, we have

$$Ax_1 + By_1 + Cz_1 + D = 0,$$
$$Ax_2 + By_2 + Cz_2 + D = 0,$$

and
$$Ax_3 + By_3 + Cz_3 + D = 0.$$

Eliminating A, B, C, D from these four equations, we have for the required equation

$$\begin{vmatrix} x, & y, & z, & 1 \\ x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \end{vmatrix} = 0.$$

17. If $S=0$ and $S'=0$ be the equations of two planes, $S - \lambda\ S' = 0$ will be the general equation of a plane through their intersection. For, since S and S' are both of the first degree, so also is $S - \lambda S'$; and hence $S - \lambda S' = 0$ represents a plane. The plane passes through all points common to $S=0$ and $S'=0$; for if the co-ordinates of any point satisfy $S=0$ and $S'=0$, those co-ordinates will also satisfy $S = \lambda S'$. Hence, since λ is arbitrary, $S - \lambda S' = 0$ is the general equation of a plane through the intersection of the given planes.

18. *To find the conditions that three planes may have a common line of intersection.*

Let the equations of the planes be

$$ax + by + cz + d = 0 \text{ ....(i)},$$
$$a'x + b'y + c'z + d' = 0 \text{ ....(ii)},$$

and
$$a''x + b''y + c''z + d'' = 0 \text{ ....(iii)}.$$

The equation of any plane through the line of intersection of (i) and (ii) is of the form

$$(ax + by + cz + d) + \lambda (a'x + b'y + c'z + d') = 0 \text{...(iv)}.$$

If the three planes have a common line of intersection, we can, by properly choosing λ, make (iv) represent the same plane as (iii). Hence corresponding coefficients must be proportional, so that

$$\frac{a+\lambda a'}{a''}=\frac{b+\lambda b'}{b''}=\frac{c+\lambda c'}{c''}=\frac{d+\lambda d'}{d''}.$$

Put each fraction equal to $-\mu$, then we have

$$a+\lambda a'+\mu a''=0,$$
$$b+\lambda b'+\mu b''=0,$$
$$c+\lambda c'+\mu c''=0,$$

and $$d+\lambda d'+\mu d''=0.$$

Eliminating λ and μ we have the required conditions, namely

$$\left\|\begin{matrix} a\ , & b\ , & c\ , & d \\ a'\ , & b'\ , & c'\ , & d' \\ a''\ , & b''\ , & c''\ , & d'' \end{matrix}\right\|=0,$$

the notation indicating that each of the four determinants, obtained by omitting one of the vertical columns, is zero.*

19. We can shew, exactly as in Conics, Art. 26, that if $Ax+By+Cz+D=0$ be the equation of a plane, and x', y', z' be the co-ordinates of any point, then $Ax'+By'+Cz'+D$ will be positive for all points on one side of the plane, and negative for all points on the other side.

20. *To find the perpendicular distance of a given point from a given plane.*

Let the equation of the given plane be

$$lx+my+nz=p \text{ ....(i)},$$

and let x', y', z' be the co-ordinates of the given point P. The equation

$$lx+my+nz=p' \text{ ....(ii)}$$

is the equation of a plane parallel to the given plane.

It will pass through the point (x', y', z') if

$$lx'+my'+nz'=p' \text{ ....(iii)}.$$

* It is easy to shew that there are only *two* independent conditions, as is geometrically obvious, for if the planes have two points in common they must have a common line of intersection.

Now if PL be the perpendicular from P on the plane (i), and ON, ON' the perpendiculars from the origin on the planes (i) and (ii) respectively, then will

$$\begin{aligned} LP &= NN' \\ &= p' - p \\ &= lx' + my' + nz' - p. \end{aligned}$$

Hence the length of the perpendicular from any point on the plane $lx + my + nz - p = 0$ is obtained by substituting the co-ordinates of the point in the expression $lx + my + nz - p$.

If the equation of the plane be $Ax + By + Cz + D = 0$, it may be written

$$\frac{A}{\sqrt{(A^2+B^2+C^2)}}x + \frac{B}{\sqrt{(A^2+B^2+C^2)}}y + \frac{C}{\sqrt{(A^2+B^2+C^2)}}z + \frac{D}{\sqrt{(A^2+B^2+C^2)}} = 0,$$

which is of the same form as (i); therefore the length of the perpendicular from (x', y', z') on the plane is

$$\frac{Ax' + By' + Cz' + D}{\sqrt{\{A^2+B^2+C^2\}}}.$$

Ex. 1. Find the equation of the plane through $(2, 3, -1)$ parallel to the plane $3x - 4y + 7z = 0$. *Ans.* $3x - 4y + 7z + 13 = 0$.

Ex. 2. Find the equation of the plane through the origin and through the intersection of the two planes $5x - 3y + 2z + 5 = 0$ and $3x - 5y - 2z - 7 = 0$. *Ans.* $25x - 23y + 2z = 0$.

Ex. 3. Shew that the three planes $2x + 5y + 3z = 0$, $x - y + 4z = 2$, and $7y - 5z + 4 = 0$ intersect in a straight line.

Ex. 4. Shew that the four planes $2x - 3y + 2z = 0$, $x + y - 3z = 4$, $3x - y + z = 2$, and $7x - 5y + 6z = 1$ meet in a point.

Ex. 5. Shew that the four points $(0, -1, -1)$ $(4, 5, 1)$, $(3, 9, 4)$ and $(-4, 4, 4,)$ lie on a plane.

Ex. 6. Are the points $(4, 1, 2)$ and $(2, 3, -1)$ on the same or on opposite sides of the plane $5x - 7y - 6z + 3 = 0$?

Ex. 7. Shew that the two points $(1, -1, 3)$ and $(3, 3, 3)$ are equidistant from the plane $5x + 2y - 7z + 9 = 0$, and on opposite sides of it.

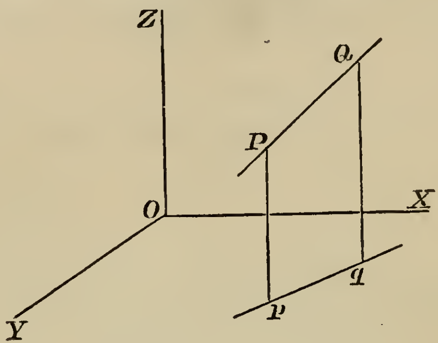
Ex. 8. Find the equations of the planes which bisect the angles between the planes $Ax + By + Cz + D = 0$, and $A'x + B'y + C'z + D' = 0$.

Ans. $\dfrac{Ax + By + Cz + D}{\sqrt{(A^2+B^2+C^2)}} = \pm \dfrac{A'x + B'y + C'z + D}{\sqrt{(A'^2+B'^2+C'^2)}}$.

Ex. 9. The locus of a point, whose distances from two given planes are in a constant ratio, is a plane.

Ex. 10. The locus of a point, which moves so that the sum of its distances from any number of fixed planes is constant, is a plane.

21. The co-ordinates of any point on the line of intersection of two planes will satisfy the equation of each of the planes. Hence any two equations of the first degree represent a straight line. We can find the equations of a straight line in their simplest form in the following manner.



Let PQ be the straight line, pq its projection on the plane XOY by lines parallel to OZ. Then the co-ordinates x and y of any point in PQ are the same as the co-ordinates x and y of its projection in pq.

Hence if $lx+my=1$ be the equation of pq, the co-ordinates of any point on PQ will satisfy the equation

$$lx+my=1.$$

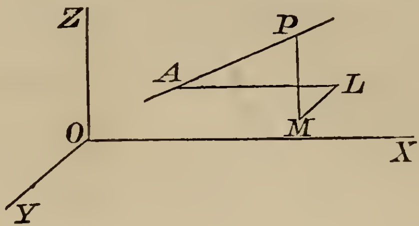
Similarly, if the equation of the projection of PQ on the plane YOZ be $ny+pz=1$, the co-ordinates of any point on PQ will satisfy the equation $ny+pz=1$. Hence the equations of the line may be written

$$lx+my=1,\ ny+pz=1.$$

It should be noticed that the equations of a straight line contain four independent constants.

The above equations are unsymmetrical and are not so useful as another form of the equations which we proceed to find.

22. Let (α, β, γ) be any point A on a straight line, and (x, y, z) any other point P on the line, at a distance r from (α, β, γ); and let l, m, n be the direction-cosines of the line.



Draw through A and P planes parallel to the co-ordinate planes so as to make a parallelopiped, and let AL, LM, MP be edges of this parallelopiped parallel to the axes of x, y, z respectively. Then AL is the projection of AP on the axis of x; therefore

$$x - \alpha = lr, \text{ or } \frac{x-\alpha}{l} = r.$$

We have similarly

$$\frac{y-\beta}{m} = r, \text{ and } \frac{z-\gamma}{n} = r.$$

Hence the equations of the line are

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r.$$

Ex. 1. To find in a symmetrical form the equations of the line of intersection of the planes $5x - 4y = 1$, $3y - 5z = 2$.

The equations may be written $\dfrac{x - \frac{1}{5}}{4} = \dfrac{y}{5} = \dfrac{z + \frac{2}{5}}{3}$. Hence the direction-cosines are proportional to 4, 5, 3. The actual values of the direction-cosines are therefore $\frac{2}{5}\sqrt{2}$, $\frac{1}{2}\sqrt{2}$, $\frac{3}{10}\sqrt{2}$.

Ex. 2. Find in a symmetrical form the equation of the line $x - 2y = 5$, $3x + y - 7z = 0$. *Ans.* $\frac{1}{2}(x-5) = y = z - \frac{15}{7}$.

Ex. 3. Find the direction-cosines of the line whose equations are $x + y - z + 1 = 0$, $4x + y - 2z + 2 = 0$. *Ans.* $\dfrac{1}{\sqrt{14}}, \dfrac{2}{\sqrt{14}}, \dfrac{3}{\sqrt{14}}$.

Ex. 4. Write down the equation of the straight line through the point (2, 3, 4) which is equally inclined to the axes. *Ans.* $x - 2 = y - 3 = z - 4$.

23. *To find the equations of a straight line through two given points.*

Let the co-ordinates of the two given points AB be x_1, y_1, z_1 and x_2, y_2, z_2; and let the co-ordinates of any point P on the line AB be x, y, z. Then the ratio of the projections of AP and AB on any axis is equal to $AP : AB$. Hence the equations of the line are

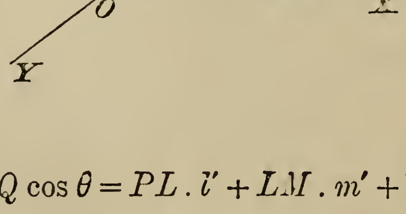
$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}.$$

24. *To find the angle between two straight lines whose direction-cosines are given.*

Let l, m, n and l', m', n' be the direction-cosines of the two lines, and let θ be the angle between them.

Let P,Q be any two points on the first line.

Draw planes through P, Q parallel to the co-ordinate planes, and let PL, LM, MQ be edges of the parallelopiped so formed. Then the projection of PQ on the second line is equal to the sum of the projections of PL, LM, and MQ on that line.



Hence $\quad PQ\cos\theta = PL \,.\, l' + LM \,.\, m' + MQ \,.\, n'.$

But $\quad PL = l \,.\, PQ, \quad LM = m \,.\, PQ,$ and $MQ = n \,.\, PQ;$

therefore $\qquad \cos\theta = ll' + mm' + nn'.$

If the lines are at right angles we have

$$ll' + mm' + nn' = 0.$$

If L, M, N are proportional to the direction-cosines of a line, the actual direction-cosines will be

$$\frac{L}{\sqrt{(L^2+M^2+N^2)}}, \quad \frac{M}{\sqrt{(L^2+M^2+N^2)}}, \quad \frac{N}{\sqrt{(L^2+M^2+N^2)}}.$$

Hence the angle between two lines whose direction-cosines are proportional to L, M, N and L', M', N' respectively is

$$\cos^{-1}\frac{LL'+MM'+NN'}{\sqrt{(L^2+M^2+N^2)}\sqrt{(L'^2+M'^2+N'^2)}}.$$

The condition of perpendicularity is as before

$$LL'+MM'+NN'=0.$$

Ex. 1. Shew that the lines $\frac{x}{1}=\frac{y}{2}=\frac{z}{1}$ and $\frac{x}{1}=\frac{y}{-1}=\frac{z}{1}$ are at right angles.

Ex. 2. Shew that the line $4x=3y=-z$ is perpendicular to the line $3x=-y=-4z$.

Ex. 3. Find the angle between the lines $\frac{x}{1}=\frac{y}{1}=\frac{z}{0}$ and $\frac{x}{3}=\frac{y}{-4}=\frac{z}{5}$.

Ans. $\cos^{-1}\frac{1}{10}$.

Ex. 4. Shew that the lines $3x+2y+z-5=0=x+y-2z-3$, and $8x-4y-4z=0=7x+10y-8z$ are at right angles.

Ex. 5. Find the acute angle between the lines whose direction-cosines are $\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{\sqrt{3}}{2}$ and $\frac{\sqrt{3}}{4}, \frac{1}{4}, -\frac{\sqrt{3}}{2}$.

Ans. 60°.

Ex. 6. Shew that the straight lines whose direction-cosines are given by the equations $2l+2m-n=0$, and $mn+nl+lm=0$ are at right angles.

Eliminating l, we have $2mn-(m+n)(2m-n)=0$, or $2m^2-mn-n^2=0$. Hence, if the direction-cosines of the two lines be l_1, m_1, n_1 and l_2, m_2, n_2, we have $\frac{m_1m_2}{n_1n_2}=-\frac{1}{2}$. Similarly $\frac{l_1l_2}{n_1n_2}=-\frac{1}{2}$. Hence the condition $l_1l_2+m_1m_2+n_1n_2=0$ is satisfied.

Ex. 7. Find the angle between the two lines whose direction-cosines are given by the equations $l+m+n=0$, $l^2+m^2-n^2=0$.

Ans. 60°.

Ex. 8. Find the equations of the straight lines which bisect the angles between the lines $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$, and $\frac{x}{l'}=\frac{y}{m'}=\frac{z}{n'}$.

Let P, Q be two points, one on each line, such that $OP=OQ=r$. Then the co-ordinates of P are lr, mr, nr, and of Q are $l'r, m'r, n'r$; hence the co-ordinates of the middle point of PQ are $\frac{1}{2}(l+l')r$, $\frac{1}{2}(m+m')r$, $\frac{1}{2}(n+n')r$. Since the middle point is on the bisector, the required equations are $\frac{x}{l+l'}=\frac{y}{m+m'}=\frac{z}{n+n'}$. Similarly the equations of the bisector of the supplementary angle are $\frac{x}{l-l'}=\frac{y}{m-m'}=\frac{z}{n-n'}$.

25. By the preceding Article

$$\cos\theta = ll' + mm' + nn';$$

therefore

$$\sin^2\theta = 1 - (ll' + mm' + nn')^2$$
$$= (l^2 + m^2 + n^2)(l'^2 + m'^2 + n'^2) - (ll' + mm' + nn')^2;$$

therefore $\sin\theta = \sqrt{\{(mn' - m'n)^2 + (nl' - n'l)^2 + (lm' - l'm)^2\}}$.

26. *To find the angle between two planes whose equations are given.*

The angle between two planes is clearly equal to the angle between two lines perpendicular to them. Now we have seen [Art. 14] that the direction-cosines of the normal to the plane

$$Ax + By + Cz + D = 0,$$

are proportional to A, B, C. Hence by Article 24 the angle between the planes whose equations are

$$Ax + By + Cz + D = 0,$$
$$A'x + B'y + C'z + D' = 0,$$

is

$$\cos^{-1}\frac{AA' + BB' + CC'}{\sqrt{(A^2 + B^2 + C^2)}\sqrt{(A'^2 + B'^2 + C'^2)}}.$$

Ex. 1. Find the equation of the plane containing the line $x+y+z=1$, $2x+3y+4z=5$, and perpendicular to the plane $x-y+z=0$.

Ans. $x-z+2=0$.

Ex. 2. At what angle do the planes $x+y+z=4$, $x-2y-z=4$ cut? Is the origin in the acute angle or in the obtuse? Is the point $(1, -3, 1)$ in the acute angle or in the obtuse?

Ans. $\cos^{-1}\frac{1}{3}\sqrt{2}$, acute, obtuse.

Ex. 3. Find the equation of the plane through $(1, 4, 3)$ perpendicular to the line of intersection of the planes $3x+4y+7z+4=0$, and $x-y+2z+3=0$; also of the plane through $(3, 1, -1)$ perpendicular to the line of intersection of the planes $3x+y-z=0$, $5x-3y+2z=0$.

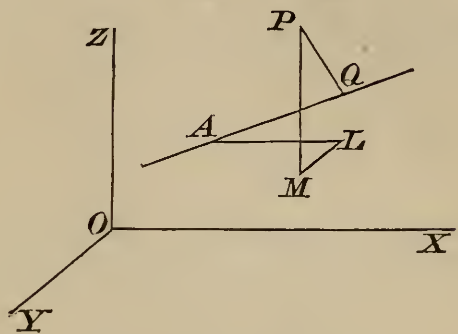
Ans. $15x+y-7z+2=0$. *Ans.* $x+11y+14z=0$.

Ex. 4. Shew that the line $\frac{x}{\lambda}=\frac{y}{\mu}=\frac{z}{\nu}$ is parallel to the plane $lx+my+nz+p=0$ if $l\lambda+m\mu+n\nu=0$, the axes being rectangular or oblique.

27. *To find the perpendicular distance of a given point from a given straight line.*

Let the equations of the line be

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}.$$



Let (f, g, h) be the given point P, and let PQ be the perpendicular from P on the line.

Let A be the point (α, β, γ), and draw through A and P planes parallel to the co-ordinate planes so as to form a parallelopiped of which AL, LM, MP are edges parallel to the axes.

Then AQ is the projection of AP on the given line, and is equal to the sum of the projections of AL, LM, and MP; therefore $\quad AQ=(f-\alpha)\,l+(g-\beta)\,m+(h-\gamma)\,n.$

Hence
$$\begin{aligned} PQ^2 &= AP^2 - AQ^2 \\ &= (f-\alpha)^2+(g-\beta)^2+(h-\gamma)^2 \\ &\qquad -\{l\,(f-\alpha)+m\,(g-\beta)+n\,(h-\gamma)\}^2. \end{aligned}$$

28. *To find the condition that two lines may intersect.*

Let the equations of the lines be

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}, \text{ and } \frac{x-\alpha'}{l'}=\frac{y-\beta'}{m'}=\frac{z-\gamma'}{n'}.$$

If the lines intersect they will lie on a plane; and, since the plane passes through (α, β, γ), we may take for its equation

$$\lambda\,(x-\alpha)+\mu\,(y-\beta)+\nu\,(z-\gamma)=0 \ldots\ldots\ldots\ldots\text{(i)}.$$

The point $(\alpha', \beta', \gamma')$ is on the plane, hence we have

$$\lambda(\alpha'-\alpha)+\mu(\beta'-\beta)+\nu(\gamma'-\gamma)=0 \quad\ldots\ldots\ldots\text{(ii)}.$$

Also, since the normal to the plane is perpendicular to both lines, we have

$$\lambda l + \mu m + \nu n = 0 \quad\ldots\ldots\ldots\ldots\ldots\text{(iii)},$$

and

$$\lambda l' + \mu m' + \nu n' = 0 \quad\ldots\ldots\ldots\ldots\ldots\text{(iv)}.$$

Eliminating λ, μ, ν from the equations (ii), (iii) and (iv) we have the required condition, namely

$$\begin{vmatrix} \alpha'-\alpha, & \beta'-\beta, & \gamma'-\gamma \\ l, & m, & n \\ l', & m', & n' \end{vmatrix} = 0.$$

If this condition be satisfied, by eliminating λ, μ, ν from (i), (iv), (iii), we find for the equation of the plane through the straight lines

$$\begin{vmatrix} x-\alpha, & y-\beta, & z-\gamma \\ l, & m, & n \\ l', & m', & n' \end{vmatrix} = 0.$$

If the equations of the lines be $a_1x+b_1y+c_1z+d_1=0$, $a_2x+b_2y+c_2z+d_2=0$, and $a_3x+b_3y+c_3z+d_3=0$, $a_4x+b_4y+c_4z+d_4=0$, the condition of intersection of the lines is the condition that the four planes may have a common point, which is found at once by eliminating x, y, z.

29. *To find the shortest distance between two straight lines whose equations are given.*

Let AKB and CLD be the given straight lines, and let KL be a line which is perpendicular to both. Then KL is the shortest distance between the given lines, for it is the projection of the line joining any other two points on the given lines[1].

Let the equations of the given lines be

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}, \text{ and } \frac{x-a'}{l'}=\frac{y-b'}{m'}=\frac{z-c'}{n'}.$$

[1] We can find KL by the following construction:—draw AE through A parallel to CD; let AP be perpendicular to the plane EAB, and let the plane PAB cut CD in L; then if LK be drawn parallel to PA it will be the line required.

Let the equations of the line on which the shortest distance lies be

$$\frac{x-\alpha}{\lambda}=\frac{y-\beta}{\mu}=\frac{z-\gamma}{\nu}\text{.....................(i)}.$$

Since the line (i) meets the given lines, we have [Art. 28]

$$\begin{vmatrix} \alpha-a, & \beta-b, & \gamma-c \\ l\,, & m\,, & n \\ \lambda\,, & \mu\,, & \nu \end{vmatrix}=0\text{ ....(ii)},$$

and
$$\begin{vmatrix} \alpha-a', & \beta-b', & \gamma-c' \\ l'\,, & m'\,, & n' \\ \lambda\,, & \mu\,, & \nu \end{vmatrix}=0\text{.........(iii)}.$$

Since (i) is perpendicular to the given lines, we have

$$\lambda l+\mu m+\nu n=0,$$

and
$$\lambda l'+\mu m'+\nu n'=0;$$

therefore
$$\frac{\lambda}{mn'-m'n}=\frac{\mu}{nl'-n'l}=\frac{\nu}{lm'-l'm}.$$

Hence, from (ii) and (iii), we see that (α, β, γ), which is an arbitrary point on the shortest distance, is on the two planes

$$\begin{vmatrix} x-a, & y-b, & z-c \\ l\,, & m\,, & n \\ mn'-m'n, & nl'-n'l, & lm'-l'm \end{vmatrix}=0,$$

and
$$\begin{vmatrix} x-a', & y-b', & z-c' \\ l'\,, & m'\,, & n' \\ mn'-m'n, & nl'-n'l, & lm'-l'm \end{vmatrix}=0.$$

These planes therefore intersect in the line on which the shortest distance lies.

We can find the length of the shortest distance from the fact that it is the projection of the line joining the points (a, b, c) and (a', b', c'). Now the projection of this line on the line whose direction-cosines are λ, μ, ν is

$$(a-a')\,\lambda+(b-b')\,\mu+(c-c')\,\nu.$$

But as above

$$\frac{\lambda}{mn' - m'n} = \frac{\mu}{nl' - n'l} = \frac{\nu}{lm' - l'm};$$

therefore each fraction is equal to

$$\frac{1}{\sqrt{\{(mn' - m'n)^2 + (nl' - n'l)^2 + (lm' - l'm)^2\}}}.$$

Hence the length of the shortest distance is

$$\frac{(a-a')(mn' - m'n) + (b-b')(nl' - n'l) + (c-c')(lm' - l'm)}{\sqrt{\{(mn' - m'n)^2 + (nl' - n'l)^2 + (lm' - l'm)^2\}}}.$$

Ex. 1. Find the perpendicular distance of an angular point of a cube from a diagonal which does not pass through that angular point.

Ans. $a\sqrt{\frac{2}{3}}$.

Ex. 2. How far is the point (4, 1, 1) from the line of intersection of $x+y+z=4$, $x-2y-z=4$? *Ans.* $\sqrt{\frac{27}{14}}$.

Ex. 3. Shew that the two lines $x-2=2y-6=3z$, $4x-11=4y-13=3z$ meet in a point, and that the equation of the plane on which they lie is $2x-6y+3z+14=0$.

Ex. 4. Find the equation of the plane through the point $(\alpha', \beta', \gamma')$, and through the line whose equations are $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$.

Ans. $\begin{vmatrix} x-\alpha, & y-\beta, & z-\gamma \\ \alpha'-\alpha, & \beta'-\beta, & \gamma'-\gamma \\ l\ , & m\ , & n \end{vmatrix} = 0.$

Ex. 5. The shortest distances between the diagonal of a rectangular parallelopiped and the edges which it does not meet are

$$\frac{bc}{\sqrt{(b^2+c^2)}}, \quad \frac{ca}{\sqrt{(a^2+c^2)}}, \quad \frac{ab}{\sqrt{(a^2+b^2)}},$$

where a, b, c are the lengths of the edges.

Ex. 6. Find the shortest distance between the straight lines

$$\tfrac{1}{2}(x-1) = \tfrac{1}{4}(y-2) = z-3, \text{ and } y - mx = z = 0.$$

Ans. $\frac{5m-10}{\sqrt{(5m^2-16m+17)}}$.

Ex. 7. Determine the length of the shortest distance between the lines $4x=3y=-z$ and $3(x-1)=-y-2=-4z+2$. Find the equations of the straight line of which the shortest distance forms a part. *Ans.* $\frac{8}{13}$.

30. If through any number of points, P, Q, R... lines be drawn either all through a fixed point, or all parallel to a fixed line; and if these lines cut a fixed plane in the points

$P', Q', R' \ldots$; then $P', Q', R' \ldots$ are called the *projections* of $P, Q, R \ldots$ on the plane. If the lines $PP', QQ', RR' \ldots$ are all perpendicular to the fixed plane, the projection is said to be *orthogonal.*

The orthogonal projection of a limited straight line on a plane is the line joining the projections of its extremities. It is easily seen that the projection of a line on a plane is equal to its length multiplied by the cosine of the angle between the line and the plane.

31. *The orthogonal projection of any plane area on any other plane is found by multiplying the area by the cosine of the angle between the planes.*

Divide the given area into a very great number of rectangles by two sets of lines parallel and perpendicular to the line of intersection of the given plane and the plane of projection. Then, those lines which are parallel to the line of intersection are unaltered by projection, and those which are perpendicular are diminished in the ratio $1 : \cos\theta$, where θ is the angle between the planes. Hence every rectangle, and therefore the sum of any number of rectangles, is diminished by projection in the ratio of $1 : \cos\theta$. But, when each of the rectangles is made indefinitely small, their sum is equal to the given area. Hence any area is diminished by projection in the ratio $1 : \cos\theta$.

32. If we have more than one plane area, we must make some convention as to the sign of the projection, and we have the following definition: the algebraic projection of any face of a polyhedron on a fixed plane is found by multiplying its area by the cosine of the angle between the normal to the fixed plane and the normal to the face, the normals to the faces being all drawn outwards or all drawn inwards.

33. Let A be the area of any plane surface; l, m, n the direction-cosines of the normal to the plane; A_x, A_y, A_z the projections of A on the co-ordinate planes. Then we have

$$A_x = l \, . \, A, \quad A_y = m \, . \, A, \quad A_z = n \, . \, A.$$

Hence, since $$l^2+m^2+n^2=1,$$
we have $$A_x^2+A_y^2+A_z^2=A^2.$$
Also the projection of A on any other plane, the direction-cosines of whose normals are l', m', n', is $A\cos\theta$; and we have

$$\begin{aligned} A\cos\theta &= (ll'+mm'+nn')\,A \\ &= l'A_x+m'A_y+n'A_z. \end{aligned}$$

Hence to find the projection of any plane area, or of the sum of any plane areas, on any given plane, we may first find the projections A_x, A_y, A_z on the co-ordinate planes, and then take the sum of the projections of A_x, A_y, A_z on the given plane.

34. *To find the volume of a tetrahedron in terms of the co-ordinates of its angular points.*

Let the co-ordinates of the angular points of the tetrahedron $ABCD$ be (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), and (x_4, y_4, z_4). The volume of a tetrahedron is one-third the area of the base multiplied by the height. Now the equation of the face BCD is

$$\begin{vmatrix} x, & y, & z, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \\ x_4, & y_4, & z_4, & 1 \end{vmatrix} = 0.$$

The perpendicular p from A on this is found by substituting the co-ordinates of A and dividing by the square root of the sum of the squares of the coefficients of x, y, and z.

Now the coefficients of x, y, z are

$$\begin{vmatrix} y_2, & z_2, & 1 \\ y_3, & z_3, & 1 \\ y_4, & z_4, & 1 \end{vmatrix}, \quad -\begin{vmatrix} x_2, & z_2, & 1 \\ x_3, & z_3, & 1 \\ x_4, & z_4, & 1 \end{vmatrix}, \quad \begin{vmatrix} x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \\ x_4, & y_4, & 1 \end{vmatrix}$$

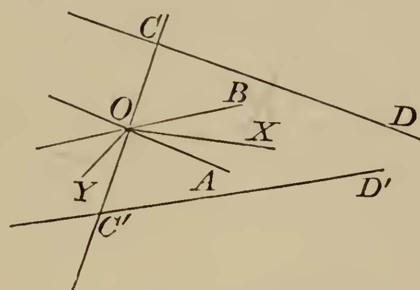
respectively; and these coefficients are respectively equal to twice the area of the projection of BCD on the planes $x=0$, $y=0$ and $z=0$. Hence the square root of the sum of the squares of the coefficients of x, y and z is, by the preceding Article, equal to $2\triangle BCD$.

Therefore $2p \,.\, \Delta BCD = \begin{vmatrix} x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_2, & 1 \\ x_4, & y_4, & z_4, & 1 \end{vmatrix}$;

therefore volume of tetrahedron

$$= \tfrac{1}{6} \begin{vmatrix} x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \\ x_4, & y_4, & z_4, & 1 \end{vmatrix}.$$

35. The equations of two straight lines can be found in a very simple form by a proper choice of axes.



Let O be the middle point of CC', the shortest distance between the two straight lines CD, $C'D'$. Through O draw OA, OB parallel to CD, $C'D'$, and let OX, OY bisect the angle AOB. Take OX, OY, OC for axes of co-ordinates; then, if AOB be 2α, the equations of OA, OB are $y = x \tan \alpha$ $z = 0$, and $y = -x \tan \alpha, z = 0$.

Hence the equations of the parallel lines CD, $C'D'$ are $y = x \tan \alpha, z = c$; and $y = -x \tan \alpha$, $z = -c$.

When it is not of importance that the axes should be *rectangular*, we may take OA, OB, OC for axes: the equations of CD, $C'D'$ will then be $y = 0$, $z = c$; and $x = 0$, $z = -c$. Also CC' may be *any* straight line which intersects CD and $C'D'$.

36. *Four given planes which have a common line of intersection cut any straight line in a range of constant cross ratio.*

Let any two lines meet the planes in the points P, Q, R, S and P', Q', R', S' respectively. Let O, O' be any two points on the line of intersection of the given planes, and let the line of intersection of the two planes $OPQRS$, $O'P'Q'R'S'$ meet the four given planes in P'', Q'', R'', S'' respectively. Then, from the pencil whose vertex is O, we have $\{PQRS\}=\{P''Q''R''S''\}$; and, from the pencil whose vertex is O', we have $\{P''Q''R''S''\}=\{P'Q'R'S'\}$. Hence $\{PQRS\}=\{P'Q'R'S'\}$, which proves the proposition.

37. Def. Two systems of planes, each of which has a common line of intersection, are said to be *homographic* when every four constituents of the one, and the corresponding four constituents of the other, have equal cross ratios.

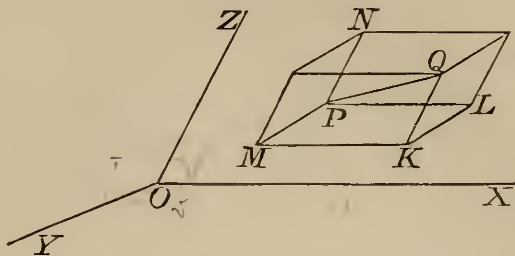
An equivalent definition [see Conics, Art. 323] is the following:—two systems of planes, each of which has a common line of intersection, are said to be homographic which are so connected that to each plane of the one system corresponds one plane, and only one, of the other.

Oblique Axes.

38. Some of the preceding investigations apply equally whether the axes are rectangular or oblique. These may be easily recognised. We proceed to consider some cases in which the formulae for oblique and rectangular axes are different.

39. Let P, Q be two points on a straight line, and through P, Q draw planes parallel to the co-ordinate planes so as to form a parallelopiped, and let PL, LK, KQ be edges parallel to the axes. Then the ratios of PL, LK, KQ to PQ are called the *direction-ratios* of the line PQ. It is clear that the direction of a line is determined by its direction-ratios.

40. *To find the angles a line makes with the axes of co-ordinates, in terms of its direction-ratios.*



Let λ, μ, ν be the angles YOZ, ZOX, XOY respectively. Let l, m, n be the direction-ratios of the line PQ, and let α, β, γ be the angles it makes with the axes. Let PL, LK, KQ be parallel to the axes so that $PL = l \,.\, PQ$, $LK = m \,.\, PQ$, $KQ = n \,.\, PQ$, as in Art. 39. Then, since the projection of PQ on the axis of x is equal to the projection of $PLKQ$, we have

$$PQ \cos\alpha = PL + LK \cos\nu + KQ \cos\mu\,;$$

therefore $\qquad \cos\alpha = l + m\cos\nu + n\cos\mu.$

Similarly $\qquad \cos\beta = l\cos\nu + m + n\cos\lambda,$

and $\qquad \cos\gamma = l\cos\mu + m\cos\lambda + n.$

41. *To find the relation between the direction-ratios of a line.*

Project PL, LK, KQ on PQ, then we have

$$PL\cos\alpha + LK\cos\beta + KQ\cos\gamma = PQ\,;$$

therefore from Art. 40,

$$l\,(l + m\cos\nu + n\cos\mu) + m\,(l\cos\nu + m + n\cos\lambda) + n\,(l\cos\mu + m\cos\lambda + n) = 1$$

or $l^2 + m^2 + n^2 + 2mn\cos\lambda + 2nl\cos\mu + 2lm\cos\nu = 1 \ldots \text{(i)}$,
which is the required relation.

Let the co-ordinates of the points P, Q be

$$x_1, y_1, z_1 \text{ and } x_2, y_2, z_2.$$

Then $\quad l\,.\,PQ = PL = x_2 - x_1,\quad m\,.\,PQ = LK = y_2 - y_1,$

and $\quad n\,.\,PQ = KQ = z_2 - z_1.$

Hence from (i) we have

$$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 + 2(y_2 - y_1)(z_2 - z_1)\cos\lambda$$
$$+ 2(z_2 - z_1)(x_2 - x_1)\cos\mu + 2(x_2 - x_1)(y_2 - y_1)\cos\nu \ldots\ldots \quad \text{(ii)},$$

which gives the distance between two points in terms of their oblique co-ordinates.

42. *To find the angle between two lines whose direction-ratios are given.*

Let l, m, n and l', m', n' be the direction-ratios of the lines PQ and $P'Q'$, and let θ be the angle between them.

Let PL, LK, KQ be parallel to the axes, so that

$$PL = l\,.\,PQ,\ LK = m\,.\,PQ, \text{ and } KQ = n\,.\,PQ.$$

Project PQ and $PLKQ$ on the line $P'Q'$; then

$$PQ\cos\theta = l\,PQ\,.\cos\alpha' + m\,PQ\,.\cos\beta' + n\,PQ\,.\cos\gamma',$$

where α', β', γ' are the angles the line $P'Q'$ makes with the axes. Hence, from Art. 40, we have

$$\begin{aligned}\cos\theta &= l\ (l' + m'\cos\nu + n'\cos\mu)\\ &\quad + m\,(l'\cos\nu + m' + n'\cos\lambda)\\ &\quad + n\ (l'\cos\mu + m'\cos\lambda + n')\\ &= ll' + mm' + nn' + (mn' + m'n)\cos\lambda + (nl' + n'l)\cos\mu\\ &\quad + (lm' + l'm)\cos\nu.\end{aligned}$$

43. *To find the volume of a tetrahedron in terms of three edges which meet in a point and of the angles they make with one another.*

Take the axes along the three edges, and let a, b, c be the lengths of the edges, and λ, μ, ν the angles they make with one another. Then

$$\text{Volume} = \tfrac{1}{6}\,abc\sin\nu\cos\theta,$$

where θ is the angle between OZ and the normal to the plane XOY.

Let the direction-ratios of the normal to the plane XOY be l, m, n. Then from Art. 40 we have

$$l + m\cos\nu + n\cos\mu = 0,$$
$$l\cos\nu + m + n\cos\lambda = 0,$$
$$l\cos\mu + m\cos\lambda + n = \cos\theta.$$

Multiply by l, m, n and add, then, from (i) Art 41,

$$n\cos\theta = 1.$$

The elimination of l, m, n from the above equations gives

$$\begin{vmatrix} 1, & \cos\nu, & \cos\mu, & 0 \\ \cos\nu, & 1, & \cos\lambda, & 0 \\ \cos\mu, & \cos\lambda, & 1, & \cos\theta \\ 0, & 0, & \cos\theta, & 1 \end{vmatrix} = 0;$$

therefore $\sin^2\nu\cos^2\theta = \begin{vmatrix} 1, & \cos\nu, & \cos\mu \\ \cos\nu, & 1, & \cos\lambda \\ \cos\mu, & \cos\lambda, & 1 \end{vmatrix}$

$$= 1 - \cos^2\lambda - \cos^2\mu - \cos^2\nu + 2\cos\lambda\cos\mu\cos\nu.$$

Hence the volume required

$$= \tfrac{1}{6}abc\sqrt{(1 - \cos^2\lambda - \cos^2\mu - \cos^2\nu + 2\cos\lambda\cos\mu\cos\nu)}.$$

Transformation of Co-ordinates.

44. *To change the origin of co-ordinates without changing the direction of the axes.*

Let f, g, h be the co-ordinates of the new origin referred to the original axes. Let P be any point whose co-ordinates referred to the original axes are x, y, z, and referred to the new axes x', y', z'. Let PL be parallel to the axis of x and let it meet YOZ in L, and $Y'OZ'$ in L'.

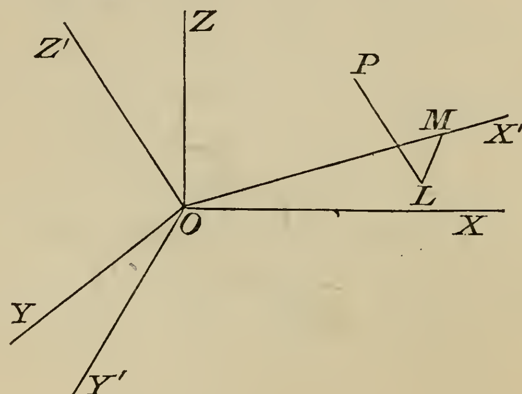
Then $$LP = x,\ L'P = x';$$
therefore $$x - x' = LL' = f.$$

Similarly $$y - y' = g,$$
and $$z - z' = h.$$

Hence, if in the equation of any surface we write $x+f$, $y+g$, $z+h$ for x, y, z respectively, we obtain the equation referred to the point (f, g, h) as origin.

45. *To change the direction of the axes without changing the origin, both systems being rectangular.*

Let l_1, m_1, n_1; l_2, m_2, n_2; and l_3, m_3, n_3 be the direction-cosines of the new axes referred to the old.



Let P be any point whose co-ordinates in the two systems are x, y, z and x', y', z'.

Draw PL perpendicular to the plane $X'OY'$ and LM perpendicular to OX'; then $OM = x'$, $ML = y'$, and $LP = z'$.

Since the projection of OP on OX is equal to the sum of the projections of OM, ML and LP, we have
$$x = l_1 x' + l_2 y' + l_3 z'.$$

Similarly $$y = m_1 x' + m_2 y' + m_3 z',$$
and $$z = n_1 x' + n_2 y' + n_3 z'.$$

These are the formulae required.

Since l_1, m_1, n_1; l_2, m_2, n_2; and l_3, m_3, n_3 are direction-cosines, we have

$$\left.\begin{array}{l} l_1^2+m_1^2+n_1^2=1 \\ l_2^2+m_2^2+n_2^2=1 \\ l_3^2+m_3^2+n_3^2=1 \end{array}\right\}.$$

Also, since OX', OY', OZ' are two and two at right angles, we have

$$\left.\begin{array}{l} l_2l_3+m_2m_3+n_2n_3=0, \\ l_3l_1+m_3m_1+n_3n_1=0, \\ l_1l_2+m_1m_2+n_1n_2=0 \end{array}\right\}.$$

and

The six relations between the nine direction-cosines which we have found above are equivalent to the following:

$$\left.\begin{array}{l} l_1^2+l_2^2+l_3^2=1, \\ m_1^2+m_2^2+m_3^2=1, \\ n_1^2+n_2^2+n_3^2=1, \end{array}\right\}$$

$$\left.\begin{array}{l} m_1n_1+m_2n_2+m_3n_3=0, \\ n_1l_1+n_2l_2+n_3l_3=0, \\ l_1m_1+l_2m_2+l_3m_3=0, \end{array}\right\}.$$

This follows at once from the fact that l_1, l_2, l_3; m_1, m_2, m_3; and n_1, n_2, n_3 are the direction-cosines of OX, OY, OZ referred to the rectangular axes OX', OY', OZ'.

46. Since

$$l_1l_2+m_1m_2+n_1n_2=0,$$

and

$$l_1l_3+m_1m_3+n_1n_3=0,$$

we have

$$\frac{l_1}{m_2n_3-m_3n_2}=\frac{m_1}{n_2l_3-n_3l_2}=\frac{n_1}{l_2m_3-l_3m_2}.$$

Hence each fraction is equal to

$$\frac{\sqrt{(l_1^2+m_1^2+n_1^2)}}{\sqrt{\{(m_2n_3-m_3n_2)^2+(n_2l_3-n_3l_2)^2+(l_2m_3-l_3m_2)^2\}}}=\pm 1. \text{ [Art. 25.]}$$

Also
$$\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}$$
$$= l_1(m_2n_3 - m_3n_2) + m_1(n_2l_3 - n_3l_2) + n_1(l_2m_3 - l_3m_2)$$
$$= \pm(l_1^2 + m_1^2 + n_1^2) = \pm 1.$$

47. If in Art. 45 the new axes are oblique we still have the relations
$$x = l_1x' + l_2y' + l_3z',$$
$$y = m_1x' + m_2y' + m_3z',$$
$$z = n_1x' + n_2y' + n_3z'.$$
We can deduce the values of x', y', z' in terms of x, y, z: the results are
$$x' \begin{vmatrix} l_1, & l_2, & l_3 \\ m_1, & m_2, & m_3 \\ n_1, & n_2, & n_3 \end{vmatrix} = \begin{vmatrix} l_2, & l_3, & x \\ m_2, & m_3, & y \\ n_2, & n_3, & z \end{vmatrix},$$
and two similar equations.

48. *The degree of an equation is unaltered by any transformation of axes.*

From the preceding Articles we see that, however the axes may be changed, the new equation is obtained by substituting for x, y, z expressions of the form $lx + my + nz + p$. These expressions are of the first degree, and therefore if they replace x, y, and z in the equation, the degree of the equation will not be *raised*. Neither can the degree of the equation be *lowered;* for, if it were, by returning to the original axes, and therefore to the original equation, the degree would be raised.

49. We shall conclude this chapter by the solution of some examples.

(1) *A line of constant length has its extremities on two fixed straight lines; shew that the locus of its middle point is an ellipse.*

If we take the axes of co-ordinates as in Art. 35, the equations of the lines will be $y = mx$, $z = c$; and $y = -mx$, $z = -c$. Let the co-ordinates of the extremities of the line in any one of its possible positions be x_1, y_1, z_1 and x_2, y_2, z_2; and let (x, y, z) be the co-ordinates of the middle point of the line. Then, if $2l$ be the length of the line, we have

$$4l^2=(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2.$$

But, since $y_1=mx_1$ and $z_1=c$, and $y_2=-mx_2$, $z_2=-c$, we have

$$x_1-x_2=\frac{1}{m}(y_1+y_2)=\frac{2y}{m},$$

$$y_1-y_2=m(x_1+x_2)=2mx,$$

$$z_1-z_2=2c, \text{ and } 2z=z_1+z_2=0.$$

Hence the locus of the middle point is the ellipse whose equations are

$$z=0,\ l^2=\frac{y^2}{m^2}+m^2x^2+c^2.$$

(2) *A line moves so as always to intersect three given straight lines, which are not all parallel to the same plane; find the equation of the surface generated by the straight line.*

Draw through each of the lines planes parallel to the other two; a parallelopiped is thus formed of which the given lines are edges. Take the centre of the parallelopiped for origin, and axes parallel to the edges, then the equations of the given lines are $y=b$, $z=-c$; $z=c$, $x=-a$; and $x=a$, $y=-b$ respectively.

Let the equations of the moving line be

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}.$$

Since this meets each of the given lines we have

$$\frac{b-\beta}{m}=\frac{-c-\gamma}{n},\quad \frac{c-\gamma}{n}=\frac{-a-\alpha}{l},\ \text{ and } \frac{a-\alpha}{l}=\frac{-b-\beta}{m}.$$

Hence, by multiplying corresponding members of the three equations, we see that (α, β, γ), an arbitrary point on the moving line, is on the surface whose equation is

$$(a-x)(b-y)(c-z)+(a+x)(b+y)(c+z)=0,$$

or

$$\frac{yz}{bc}+\frac{zx}{ca}+\frac{xy}{ab}+1=0.$$

(3) *The lines of intersection of corresponding planes of two homographic systems describe a surface of the second degree.*

We may take $y=mx$, $z=c$, and $y=-mx$, $z=-c$ for the equations of the lines of intersection of the two systems of planes [see Art. 35.]

Let the equations of corresponding planes of the two systems be

$$y-mx+\lambda(z-c)=0,$$

and

$$y+mx+\lambda'(z+c)=0.$$

Since the systems are homographic there is one value of λ' for every value of λ, and one value of λ for every value of λ'; hence λ, λ' must be connected by a relation of the form

$$\lambda\lambda'+A\lambda+B\lambda'+C=0.$$

Substitute for λ and λ', and we have

$$y^2 - m^2x^2 - A(z+c)(y-mx) - B(z-c)(y+mx) + C(z^2-c^2) = 0.$$

Hence the line of intersection of corresponding planes describes a surface of the second degree.

Examples on Chapter II.

1. If P be a fixed point on a straight line through the origin equally inclined to the three axes of co-ordinates, any plane through P will intercept lengths on the co-ordinate axes the sum of whose reciprocals is constant.

2. Shew that the six planes, each passing through one edge of a tetrahedron and bisecting the opposite edge, meet in a point.

3. Through the middle point of every edge of a tetrahedron a plane is drawn perpendicular to the opposite edge; shew that the six planes so drawn will meet in a point.

4. The equation of the plane through $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$, and which is perpendicular to the plane containing $\frac{x}{m} = \frac{y}{n} = \frac{z}{l}$ and $\frac{x}{n} = \frac{y}{l} = \frac{z}{m}$, is $x(m-n) + y(n-l) + z(l-m) = 0$.

5. Shew that the straight lines

$$\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma}, \quad \frac{x}{a\alpha} = \frac{y}{b\beta} = \frac{z}{c\gamma}, \quad \frac{x}{l} = \frac{y}{m} = \frac{z}{n},$$

will lie in one plane, if

$$\frac{l}{\alpha}(b-c) + \frac{m}{\beta}(c-a) + \frac{n}{\gamma}(a-b) = 0.$$

6. Two systems of rectangular axes have the same origin; if a plane cut them at distances a, b, c, and a', b', c' from the origin, then

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2}.$$

7. Determine the locus of a point which moves so as always to be equally distant from two given straight lines.

8. Through two straight lines given in space two planes are drawn at right angles to one another; find the locus of their line of intersection.

9. A line of constant length has its extremities on two given straight lines; find the equation of the surface generated by it, and shew that any point in the line describes an ellipse.

10. Shew that the two straight lines represented by the equations $ax + by + cz = 0$, $yz + zx + xy = 0$ will be perpendicular if

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0.$$

11. Find the plane on which the area of the projection of the hexagon, formed by six edges of a cube which do not meet a given diagonal, is a maximum.

12. Prove that the four planes

$$my + nz = 0,\ nz + lx = 0,\ lx + my = 0,\ lx + my + nz = p,$$

form a tetrahedron whose volume is $\dfrac{2p^3}{3lmn}$.

13. Find the surface generated by a straight line which is parallel to a fixed plane and meets two given straight lines.

14. A straight line meets two given straight lines and makes the same angle with both of them; find the surface which it generates.

15. Any two finite straight lines are divided in the same ratio by a straight line; find the equation of the surface which it generates.

16. A straight line always parallel to the plane of yz passes through the curves $x^2 + y^2 = a^2$, $z = 0$, and $x^2 = az$, $y = 0$; prove that the equation of the surface generated is

$$x^4y^2 = (x^2 - az)^2 (a^2 - x^2).$$

17. Three straight lines mutually at right angles meet in a point P, and two of them intersect the axes of x and y respectively, while the third passes through a fixed point $(0, 0, c)$ on the axis of z. Shew that the equation of the locus of P is

$$x^2 + y^2 + z^2 = 2cz.$$

18. Find the surface generated by a straight line which meets

$y = mx$, $z = c$; $y = -mx$, $z = -c$; and $y^2 + z^2 = c^2$, $x = 0$.

19. P, P' are points on two fixed non-intersecting straight lines AB, $A'B'$ such that the rectangle AP, $A'P'$ is constant. Find the surface generated by the line PP'.

20. Find the condition that

$$ax^2 + by^2 + cz^2 + 2a'yz + 2b'zx + 2c'xy = 0$$

may represent a pair of planes; and supposing it satisfied, if θ be the angle between the planes, prove that

$$\tan\theta = \frac{2\sqrt{a'^2 + b'^2 + c'^2 - bc - ca - ab}}{a + b + c}.$$

21. Find the volume of the tetrahedron formed by planes whose equations are $y + z = 0$, $z + x = 0$, $x + y = 0$, and $x + y + z = 1$.

22. Find the volume of a tetrahedron, having given the equations of its plane faces.

23. Shew that the sum of the projections of the faces of a closed polyhedron on any plane is zero.

24. Find the co-ordinates of the centre of the sphere inscribed in the tetrahedron formed by the planes whose equations are $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.

25. Find the co-ordinates of the centre of the sphere inscribed in the tetrahedron formed by the planes whose equations are $y + z = 0$, $z + x = 0$, $x + y = 0$, and $x + y + z = a$.

1. Find Princ. planes.
2. " center
3. " Equat. of cone.

CHAPTER III.

SURFACES OF THE SECOND DEGREE.

50. The most general equation of the second degree, viz.
$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0,$$
contains ten constants. But, since we may multiply or divide the equation by any constant quantity without altering the relation between x, y, and z which it indicates, there are really only nine constants which are fixed for any particular surface, viz. the nine ratios of the ten constants a, b, c, &c. to one another. A surface of the second degree can therefore be made to satisfy nine conditions and no more. The nine conditions which a surface of the second degree can satisfy must be such that each gives rise to one relation among the constants, as, for instance, the condition of passing through a given point. Such conditions as give two or more relations between the constants must be reckoned as two or more of the nine.

We shall throughout the present chapter assume that the equation of the second degree is of the above form, unless it is otherwise expressed. The left-hand side of the equation will be sometimes denoted by $F(x, y, z)$.

51. *To find the points where a given straight line cuts the surface represented by the general equation of the second degree.*

[illegible] tan.

Let the equations of the straight line be

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r.$$

To find the points common to this line and the surface, we have the equation

$$a(\alpha+lr)^2+b(\beta+mr)^2+c(\gamma+nr)^2+2f(\beta+mr)(\gamma+nr)$$
$$+2g(\gamma+nr)(\alpha+lr)+2h(\alpha+lr)(\beta+mr)+2u(\alpha+lr)$$
$$+2v(\beta+mr)+2w(\gamma+nr)+d=0,$$

or

$$r^2(al^2+bm^2+cn^2+2fmn+2gnl+2hlm)+r\left\{l\frac{dF}{d\alpha}+m\frac{dF}{d\beta}+n\frac{dF}{d\gamma}\right\}$$
$$+F(\alpha,\beta,\gamma)=0\text{............(i)}.$$

Since this is a quadratic equation, any straight line meets the surface in *two* points.

Hence all straight lines which lie in any particular plane meet the surface in two points. So that, *all plane sections of a surface of the second degree are conics.*

In what follows surfaces of the second degree will generally be called *conicoids.*

52. *To find the equation of the tangent plane at any point of a conicoid.*

If (α, β, γ) be a point on $F(x, y, z)=0$, one root of the equation found in the preceding Article will be zero. Two roots will be zero if l, m, n satisfy the relation

$$l\frac{dF}{d\alpha}+m\frac{dF}{d\beta}+n\frac{dF}{d\gamma}=0\text{.................(i)}.$$

The line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ will in that case be a tangent line to the surface, the point of contact being (α, β, γ).

If we eliminate l, m, n between the equations of the line, and the equation (i), we see that all the tangent lines lie in the plane whose equation is

$$(x-\alpha)\frac{dF}{d\alpha}+(y-\beta)\frac{dF}{d\beta}+(z-\gamma)\frac{dF}{d\gamma}=0\text{....(ii)}.$$

This plane is called the *tangent plane* at the point (α, β, γ).

If we write the equation (ii) in full, we obtain

$$x(a\alpha + h\beta + g\gamma + u) + y(h\alpha + b\beta + f\gamma + v) + z(g\alpha + f\beta + c\gamma + w)$$
$$= a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta + u\alpha + v\beta + w\gamma.$$

Add $u\alpha + v\beta + w\gamma + d$ to both sides, then the right side becomes $F(\alpha, \beta, \gamma)$, which is zero; we therefore have for the equation of the tangent plane at (α, β, γ)

$$x(a\alpha + h\beta + g\gamma + u) + y(h\alpha + b\beta + f\gamma + v) + z(g\alpha + f\beta + c\gamma + w)$$
$$+ u\alpha + v\beta + w\gamma + d = 0 \ldots \text{(iii)}.$$

Ex. 1. Find the equation of the tangent plane at the point (x', y', z') on the surface $ax^2 + by^2 + cz^2 + d = 0$. *Ans.* $ax'x + by'y + cz'z + d = 0$.

Ex. 2. Find the equation of the tangent plane at the point (x', y', z') on the surface $ax^2 + by^2 + 2z = 0$. *Ans.* $ax'x + by'y + z + z' = 0$.

53. The condition that the tangent plane at (α, β, γ) may pass through a particular point (x', y', z') is

$$x'(a\alpha + h\beta + g\gamma + u) + y'(h\alpha + b\beta + f\gamma + v) + z'(g\alpha + f\beta + c\gamma + w)$$
$$+ u\alpha + v\beta + w\gamma + d = 0.$$

This condition is equivalent to

$$\alpha(ax' + hy' + gz' + u) + \beta(hx' + by' + fz' + v) + \gamma(gx' + fy' + cz' + w)$$
$$+ ux' + vy' + wz' + d = 0.$$

From the last equation we see that all the points, the tangent planes at which pass through the particular point (x', y', z'), lie on a plane, namely on the plane whose equation is

$$x(ax' + hy' + gz' + u) + y(hx' + by' + fz' + v)$$
$$+ z(gx' + fy' + cz' + w) + ux' + vy' + wz' + d = 0.$$

This plane is called the *polar plane* of the point (x', y', z').

The polar plane of any point P cuts the surface in a conic, and the line joining P to any point on this conic is a tangent line. The assemblage of such lines forms a cone, which is called the *tangent cone* from P to the conicoid.

The equation of the polar plane of the origin, found by putting $x' = y' = z' = 0$ in the above, is

$$ux + vy + wz + d = 0.$$

54. The condition that the polar plane of (x', y', z') may pass through (α, β, γ) is as above

$$\alpha(ax' + hy' + gz' + u) + \beta(hx' + by' + fz' + v)$$
$$+ \gamma(gx' + fy' + cz' + w) + ux' + vy' + wz' + d = 0.$$

This equation is unaltered if we interchange α and x', β and y', and γ and z'; it therefore follows that if the polar plane of any point P with respect to a conicoid pass through a point Q, then will the polar plane of Q pass through P.

55. Let R be any point on the line of intersection of the polar planes of P, Q.

Then, since R is on the polar plane of P and also on the polar plane of Q, the polar plane of R will pass through P and through Q, and therefore through the line PQ. Similarly the polar plane of S, any other point on the line of intersection, will pass through the line PQ.

Two lines which are such that the polar plane with respect to a conicoid of any point on the one passes through the other, are called *polar* lines, or *conjugate* lines.

56. *If any chord of a conicoid be drawn through a point O it will be cut harmonically by the surface and the polar plane of O.*

Take the point O for origin, and let the surface be given by the general equation of the second degree.

Let the equations of any line, which cuts the surface in P, Q and the polar plane of O in R, be

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} = r.$$

To find the points where the line cuts the surface we have, as in Art. 51, the quadratic equation

$$r^2(al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm)$$
$$+ 2r(ul + vm + wn) + d = 0.$$

Hence $$\frac{1}{OP} + \frac{1}{OQ} = -\frac{2}{d}(ul + vm + wn).$$

The equation of the polar plane of O is

$$ux + vy + wz + d = 0.$$

Hence $$\frac{1}{OR} = -\frac{1}{d}(ul + vm + wn);$$

therefore $$\frac{1}{OP} + \frac{1}{OQ} = \frac{2}{OR},$$

which proves the proposition.

57. *To find the condition that a given plane may touch a conicoid.*

Let the equation of the given plane be

$$lx + my + nz + p = 0 \quad \ldots\ldots\ldots\ldots\ldots(\text{i}).$$

The tangent plane at (x', y', z') is

$$x(ax' + hy' + gz' + u) + y(hx' + by' + fz' + v)$$
$$+ z(gx' + fy' + cz' + w) + ux' + vy' + wz' + d = 0 \ldots\ldots(\text{ii}).$$

If the planes represented by (i) and (ii) are the same we have

$$\frac{ax' + hy' + gz' + u}{l} = \frac{hx' + by' + fz' + v}{m} = \frac{gx' + fy' + cz' + w}{n}$$
$$= \frac{ux' + vy' + wz' + d}{p}.$$

Put each fraction equal to $-\lambda$; then we have

$$\begin{aligned} ax' + hy' + gz' + u + \lambda l &= 0, \\ hx' + by' + fz' + v + \lambda m &= 0, \\ gx' + fy' + cz' + w + \lambda n &= 0, \\ ux' + vy' + wz' + d + \lambda p &= 0. \end{aligned}$$

Also, since (x', y', z') is on the given plane,

$$lx' + my' + nz' + p = 0.$$

Eliminating x', y', z', λ, we obtain the required condition, namely

$$\begin{vmatrix} a, & h, & g, & u, & l \\ h, & b, & f, & v, & m \\ g, & f, & c, & w, & n \\ u, & v, & w, & d, & p \\ l, & m, & n, & p, & 0 \end{vmatrix} = 0.$$

The determinant when expanded is

$$Al^2 + Bm^2 + Cn^2 + Dp^2 + 2\,Fmn + 2\,Gnl + 2\,Hlm + 2\,Ulp + 2Vmp + 2Wnp = 0,$$

where A, B, C, &c. are the co-factors of a, b, c, &c. in the determinant

$$\begin{vmatrix} a, & h, & g, & u \\ h, & b, & f, & v \\ g, & f, & c, & w \\ u, & v, & w, & d \end{vmatrix}.$$

We will give special investigations in the two following cases which are of great importance:

I. Let the equation of the surface be

$$ax^2 + by^2 + cz^2 + d = 0.$$

The tangent plane at any point (x', y', z') is

$$ax'x + by'y + cz'z + d = 0.$$

Hence, comparing this equation with the given equation

$$lx + my + nz + p = 0,$$

we have $\dfrac{ax'}{l} = \dfrac{by'}{m} = \dfrac{cz'}{n} = \dfrac{d}{p}$. Each fraction is equal to

$$\frac{\sqrt{(ax'^2 + by'^2 + cz'^2 + d)}}{\sqrt{\left(\dfrac{l^2}{a} + \dfrac{m^2}{b} + \dfrac{n^2}{c} + \dfrac{p^2}{d}\right)}};$$

hence, since $\qquad ax'^2 + by'^2 + cz'^2 + d = 0,$

the required condition of tangency is

$$\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} + \frac{p^2}{d} = 0.$$

II. Let the equation of the surface be

$$ax^2 + by^2 + 2z = 0.$$

The tangent plane at any point (x', y', z') is

$$ax'x + by'y + z + z' = 0.$$

Hence, comparing this equation with the given equation

$$lx + my + nz + p = 0,$$

we have $\dfrac{ax'}{l}=\dfrac{by'}{m}=\dfrac{1}{n}=\dfrac{z'}{p}$. Each fraction is equal to

$$\frac{\sqrt{(ax'^2+by'^2+2z')}}{\sqrt{\left(\dfrac{l^2}{a}+\dfrac{m^2}{b}+2np\right)}};$$

hence, since $$ax'^2+by'^2+2z'=0,$$

the required condition of tangency is

$$\frac{l^2}{a}+\frac{m^2}{b}+2np=0.$$

58. If we find, as in Article 51, the quadratic equation giving the segments of a chord through (α, β, γ) the roots of the equation will be equal and opposite, if

$$l\frac{dF}{d\alpha}+m\frac{dF}{d\beta}+n\frac{dF}{d\gamma}=0 \text{ .... (i).}$$

In this case (α, β, γ) will be the middle point of the chord. Hence an infinite number of chords of the conicoid have the point (α, β, γ) for their middle point.

If we eliminate l, m, n between the equations of the chord and (i), we see that all such chords are in the plane whose equation is

$$(x-\alpha)\frac{dF}{d\alpha}+(y-\beta)\frac{dF}{d\beta}+(z-\gamma)\frac{dF}{d\gamma}=0 \text{ ...... (ii).}$$

Hence (α, β, γ) is the centre of the conic in which (ii) meets the surface.

This result should be compared with that obtained in Art. 52.

Ex. 1. The locus of the centre of all plane sections of a conicoid which pass through a fixed point is a conicoid.

The equation of the locus is $(f-x)\dfrac{dF}{dx}+(g-y)\dfrac{dF}{dy}+(h-z)\dfrac{dF}{dz}=0$, where f, g, h are the co-ordinates of the fixed point.

Ex. 2. The locus of the centre of parallel sections of a conicoid is a straight line.

The section whose centre is (α, β, γ) is parallel to the given plane $lx+my+nz=0$ if

$$\frac{\frac{dF}{d\alpha}}{l}=\frac{\frac{dF}{d\beta}}{m}=\frac{\frac{dF}{d\gamma}}{n}.$$

Hence the locus is the straight line whose equations are

$$\frac{1}{l}\frac{dF}{dx}=\frac{1}{m}\frac{dF}{dy}=\frac{1}{n}\frac{dF}{dz}.$$

The straight lines clearly all pass through the point of intersection of the planes $\frac{dF}{dx}=\frac{dF}{dy}=\frac{dF}{dz}=0$.

59. *To find the locus of the middle points of a system of parallel chords of a conicoid.*

As in the preceding Article, (α, β, γ) will be the middle point of the chord whose direction-cosines are l, m, n, if

$$l\frac{dF}{d\alpha}+m\frac{dF}{d\beta}+n\frac{dF}{d\gamma}=0.$$

Hence the locus of the middle points of all chords whose direction-cosines are l, m, n is the plane whose equation is

$$l\frac{dF}{dx}+m\frac{dF}{dy}+n\frac{dF}{dz}=0.$$

Def. The locus of the middle points of a system of parallel chords of a conicoid is called the *diametral plane.*

If the plane be perpendicular to the chords it bisects, it is called a *principal* plane.

60. *To find the equations of the principal planes of a conicoid.*

The diametral plane of the chords whose direction-cosines are l, m, n is

$$l\frac{dF}{dx}+m\frac{dF}{dy}+n\frac{dF}{dz}=0,$$

or, writing the equation in full,

$$l(ax+hy+gz+u)+m(hx+by+fz+v)+n(gx+fy+cz+w)=0,$$

or $$x(al+hm+gn)+y(hl+bm+fn)+z(gl+fm+cn)+ul+vm+wn=0.$$

If this plane be perpendicular to the chords it bisects, we have

$$\frac{al+hm+gn}{l}=\frac{hl+bm+fn}{m}=\frac{gl+fm+cn}{n}.$$

Put λ for the common value of these fractions, then

$$\left.\begin{array}{llll}(a-\lambda)\,l & +hm & +gn & =0,\\ hl & +(b-\lambda)\,m & +fn & =0,\\ gl & +fm & +(c-\lambda)\,n & =0.\end{array}\right\}\ldots\ldots\text{(i)}.$$

Eliminating l, m, n we have

$$\begin{vmatrix} a-\lambda, & h, & g \\ h, & b-\lambda, & f \\ g, & f, & c-\lambda \end{vmatrix}=0,$$

or $$\lambda^3-(a+b+c)\,\lambda^2+(bc+ca+ab-f^2-g^2-h^2)\,\lambda$$
$$-(abc+2fgh-af^2-bg^2-ch^2)=0.$$

This is a cubic equation for determining λ; and when λ is determined, any two of the three equations (i) will give the corresponding values of l, m, n.

Since *one* root of a cubic is always real, it follows that there is always *one* principal plane.

Find the principal planes of the following surfaces:

(i) $x^2+y^2-z^2+2yz+2zx-2xy=a^2$.

(ii) $11x^2+10y^2+6z^2-8yz+4zx-12xy=1$.

Ans. (i) $x+y+z=0$, $x-y=0$, $x+y-2z=0$.

Ans. (ii) $x+2y+2z=0$, $2x+y-2z=0$, $2x-2y+z=0$.

61. *All parallel plane sections of a conicoid are similar and similarly situated conics.*

Change the axes of co-ordinates in such a way that the plane of xy may be one of the system of parallel planes; and let the equation of the surface be the general equation of the second degree.

Let the equation of any one of the planes be $z=k$. At all points of the section of the surface $F(x, y, z)=0$, by the plane $z = k$ both these relations are satisfied; we therefore have

$$ax^2 + by^2 + ck^2 + 2fyk + 2gkx + 2hxy + 2ux + 2vy + 2wk + d = 0 \ldots\ldots\ldots\ldots(\text{i}).$$

Now the equation (i) represents a cylinder whose generating lines are parallel to the axis of z, and which is cut by the plane $z = 0$ in the curve represented by (i).

Since parallel sections of a cylinder are similar and similarly situated curves, the section of the surface $F(x, y, z) = 0$ by $z = k$ is similar to the conic represented by (i) and $z = 0$; and all such conics, for different values of k, are clearly similar and similarly situated: this proves the proposition.

Classification of Conicoids.

62. We proceed to find the nature of the different surfaces whose equations are of the second degree; and we will first shew that we can always change the directions of the axes of co-ordinates in such a way that the coefficients of yz, zx, and xy in the transformed equation are all zero.

63. We have seen [Art. 60] that there is at least *one* diametral plane which is perpendicular to the chords it bisects.

Take this plane for the plane $z = 0$ in a new system of co-ordinates.

The degree of the equation of the surface will not be altered by the transformation; hence the equation will be of the form

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0.$$

By supposition the plane $z = 0$ bisects all chords parallel to the axis of z; therefore if (x', y', z') be *any* point on the surface, the point $(x', y', -z')$ will also be on the surface. From this we see at once that $f = g = w = 0$.

Now turn the axes through an angle $\frac{1}{2}\tan^{-1}\dfrac{2h}{a-b}$, then [See Conics, Art. 167] the term involving xy will disappear.

Hence we have reduced the equation to a form in which the terms yz, zx, and xy are all absent.

64. When the terms yz, zx, xy are all absent from the equation of a conicoid, it follows from Art. 60 that the co-ordinate planes are all parallel to principal planes. Hence by the preceding article, there are always *three* principal planes, which are two and two at right angles. This shews that all the roots of the cubic equation found in Art. 60 are real.

For an algebraical proof of this important theorem see Todhunter's *Theory of Equations*.

65. We have seen that the general equation of the second degree can in all cases be reduced to the form

$$Ax^2 + By^2 + Cz^2 + 2Ux + 2Vy + 2Wz + D = 0.$$

I. Let A, B, C be all finite.

We can then write the equation

$$A\left(x + \frac{U}{A}\right)^2 + B\left(y + \frac{V}{B}\right)^2 + C\left(z + \frac{W}{C}\right)^2 = \frac{U^2}{A} + \frac{V^2}{B} + \frac{W^2}{C} - D \equiv D'.$$

Hence, by a change of origin, we have

$$Ax^2 + By^2 + Cz^2 = D'.$$

If D' be not zero we have

$$\frac{x^2}{\frac{D'}{A}} + \frac{y^2}{\frac{D'}{B}} + \frac{z^2}{\frac{D'}{C}} = 1,$$

which we can write in the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{..................} (\alpha),$$

or $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \text{..................} (\beta),$$

or $$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \text{..................} (\gamma),$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1$$

according as $\frac{D'}{A}$, $\frac{D'}{B}$, $\frac{D'}{C}$ are all positive, two positive and one negative, or one positive and two negative. [If all three are negative the surface is clearly imaginary.]

If D' be zero, we have

$$Ax^2 + By^2 + Cz^2 = 0 \;\ldots\ldots\ldots\ldots\ldots(\delta).$$

II. Let C, any one of the three coefficients A, B, C, be zero.

Write the equation in the form

$$A\left(x+\frac{U}{A}\right)^2 + B\left(y+\frac{V}{B}\right)^2 + 2Wz + D - \frac{U^2}{A} - \frac{V^2}{B} = 0;$$

then, if W be not zero, the equation can, by a change of origin, be reduced to

$$Ax^2 + By^2 + 2Wz = 0 \;\ldots\ldots\ldots\ldots\ldots(\epsilon).$$

If W be zero, we have the form

$$Ax^2 + By^2 + D' = 0 \;\ldots\ldots\ldots\ldots\ldots(\zeta),$$

or, if D' be zero, the form

$$Ax^2 + By^2 = 0 \;\ldots\ldots\ldots\ldots\ldots(\eta).$$

III. Let B, C, two of the three coefficients, be zero.

We then have

$$A\left(x+\frac{U}{A}\right)^2 + 2Vy + 2Wz + D - \frac{U^2}{A} = 0.$$

Now take $2Vy + 2Wz + D - \frac{U^2}{A} = 0$ for the plane $y = 0$, and the equation reduces to the form

$$x^2 = 2ky \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\theta).$$

If however $V = W = 0$, the equation is equivalent to

$$x^2 = k' \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\iota).$$

66. We now proceed to consider the nature of the surfaces whose equations are (α), (β),.....(ι); to one of which forms we have seen that the general equation is reducible.

The surface whose equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

is called an *ellipsoid.*

Let a, b, c be in descending order of magnitude; then (x, y, z) being any point on the surface, we have

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{a^2} \ngtr 1,$$

and $$\frac{x^2}{c^2}+\frac{y^2}{c^2}+\frac{z^2}{c^2} \nless 1.$$

So that no point on the surface is at a distance from the origin greater than a, or less than c. The surface is therefore limited in every direction; and, since all plane sections of a conicoid are conics, it follows that all plane sections of an ellipsoid are ellipses.

The surface is clearly symmetrical about each of the co-ordinate planes.

If r be the length of a semi-diameter whose direction-cosines are l, m, n, we have the relation

$$\frac{1}{r^2}=\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}.$$

If two of the coefficients are equal, b and c suppose, the section by the plane $x=0$, and therefore [Art. 61] by any plane parallel to $x=0$, is a circle. Hence the surface is that formed by the revolution of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about the axis of x.

The surface formed by the revolution of an ellipse about its major axis is called a *prolate spheroid;* that formed by the revolution about the minor axis is called an *oblate spheroid.*

If $a=b=c$ the equation of the surface is $x^2+y^2+z^2=a^2$, which from Art. 5 represents a sphere.

67. The surface whose equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,$$

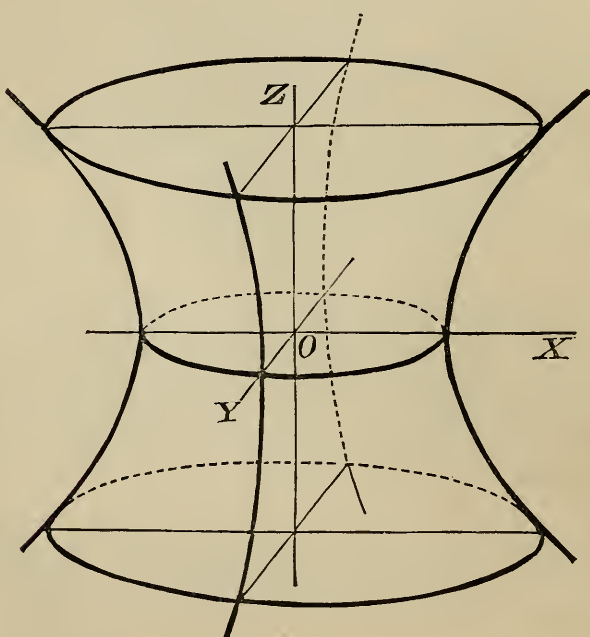
is called an *hyperboloid of one sheet.*

The intercepts on the axes of x and y are real, and those on the axis of z are imaginary.

The surface is clearly symmetrical about each of the co-ordinate planes.

The sections by the planes $x=0$ and $y=0$ are hyperbolas, and that by $z=0$ is an ellipse.

The section by $z=k$ is also an ellipse, the projection of which on $z=0$ is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1+\frac{k^2}{c^2}$, and the section becomes greater and greater as k becomes greater and greater.



If $a=b$, the section of the surface by any plane parallel to $z=0$ is a circle. Hence the surface is that formed by the revolution of the hyperbola $\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$ about its conjugate axis.

The figure shews the nature of the surface.

68. The surface whose equation is

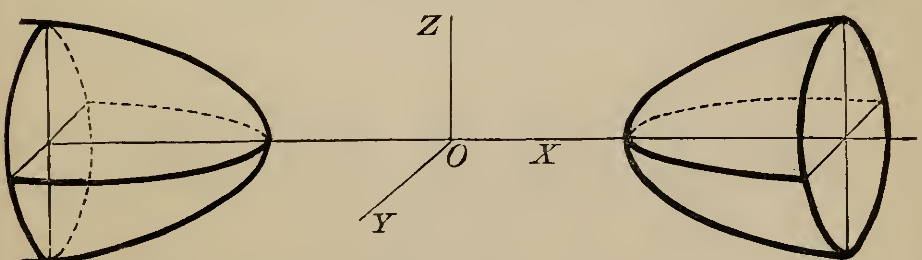
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

is called an *hyperboloid of two sheets*.

The intercepts on the axis of x are real, those on the other two axes are imaginary.

The sections by the planes $y = 0$ and $z = 0$ are hyperbolas.

The section by the plane $x = 0$ is imaginary. The parallel plane $x = k$ does not meet the surface in real points unless $k^2 > a^2$. If $k^2 > a^2$ the section is an ellipse the axes of which become greater and greater as k becomes greater and greater. The surface therefore consists of two detached portions as in the figure.



If $b = c$, the section by any plane parallel to $x = 0$ is a circle. Hence the surface is that formed by the revolution of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ about its transverse axis.

69. The surface whose equation is $Ax^2 + By^2 + Cz^2 = 0$ is a *cone*.

A *cone* is a surface generated by straight lines which always pass through a fixed point, and which obey some other law. The lines are called generating lines, and the fixed point through which they pass is called the vertex of the cone.

If the vertex of a cone be taken as origin, the equation of the surface is homogeneous. This follows at once from the consideration that if (x, y, z) be any point P on the surface, any other point (kx, ky, kz) on the line OP is also on the surface.

Conversely any homogeneous equation represents a cone whose vertex is the origin of co-ordinates. For, if the values $x, y, z,$ satisfy a homogeneous equation, so also will $kx, ky, kz,$ whatever the value of k may be. Hence the line through the origin and any point on the surface lies wholly on the surface.

The general equation of a cone of the second degree, or quadric cone, referred to its vertex as origin is therefore

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0.$$

70. If r be the length of the semi-diameter of the surface $ax^2 + by^2 + cz^2 = 1$, we have the relation

$$\frac{1}{r^2} = al^2 + bm^2 + cn^2.$$

Hence the direction-cosines of the lines which meet the surface at an infinite distance satisfy the relation

$$al^2 + bm^2 + cn^2 = 0.$$

Such lines are therefore generating lines of the cone

$$ax^2 + by^2 + cz^2 = 0.$$

This cone is called the *asymptotic cone* of the surface.

71. The equation $Ax^2 + By^2 + 2Wz = 0$ is equivalent to $\frac{x^2}{l} + \frac{y^2}{l'} = 2z$, or $\frac{x^2}{l} - \frac{y^2}{l'} = 2z$, according as the signs of A and B are alike or different.

The surface whose equation is

$$\frac{x^2}{l}+\frac{y^2}{l'}=2z,$$

is called an *elliptic paraboloid.*

The sections by the planes $x=0$ and $y=0$ are parabolas having a common axis, and whose concavities are in the same direction.

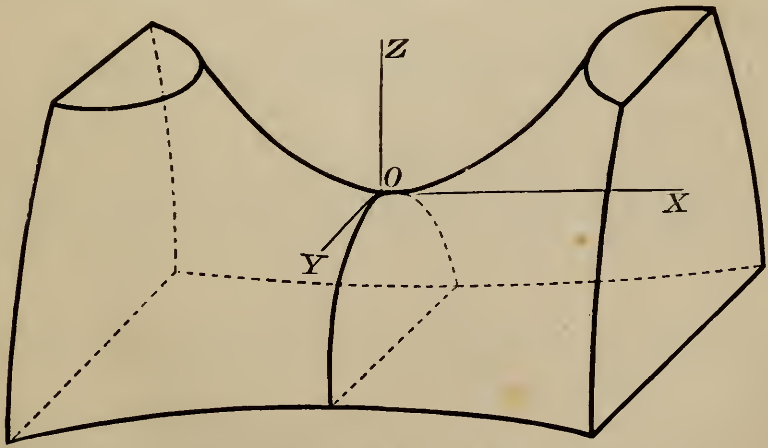
The section by any plane parallel to $z=0$ is an ellipse if the plane be on the positive side of $z=0$, and is imaginary if the plane be on the negative side of $z=0$. Hence the surface is entirely on the positive side of the plane $z=0$, and extends to an infinite distance.

The surface whose equation is

$$\frac{x^2}{l}-\frac{y^2}{l'}=2z,$$

is called an *hyperbolic paraboloid.*

The sections by the planes $x=0$ and $y=0$ are parabolas which have a common axis, and whose concavities are in opposite directions.



The surface is on both sides of the plane $z=0$, and extends to an infinite distance in both directions.

The section by the plane $z=0$ is the two straight lines given by the equation $\frac{x^2}{l}-\frac{y^2}{l'}=0$. The section by any plane parallel to $z=0$ is an hyperbola: on one side of the plane $z=0$ the real axis of the hyperbola is parallel to the axis of x, and on the other side the real axis is parallel to the axis of y.

The figure shews the nature of the surface.

72. It is important to notice that the elliptic paraboloid is a limiting form of the ellipsoid, or of the hyperboloid of two sheets; and that the hyperbolic paraboloid is a limiting form of the hyperboloid of one sheet.

This can be shewn in the following manner.

The equation of the ellipsoid referred to $(-a, 0, 0)$ as origin is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-\frac{2x}{a}=0$. Now suppose that a, b, c all become infinite, while $\frac{b^2}{a}$, $\frac{c^2}{a}$ remain finite and equal respectively to l and l'; then, in the limit, we have $\frac{y^2}{l}+\frac{z^2}{l'}=2x$, which is the equation of an elliptic paraboloid.

The other cases can be proved in a similar manner.

73. The equation $Ax^2+By^2+D=0$ represents a cylinder [Art. 10], being a hyperbolic cylinder if A and B have different signs, and an elliptic cylinder if A and B have the same sign. If the signs of A, B, D are all the same the surface is imaginary.

The equation $Ax^2+By^2=0$ represents two intersecting planes, which are imaginary or real according as the signs of A and B are alike or different.

The equation $x^2=2ky$ represents a cylinder whose guiding curve is a parabola, and which is called a parabolic cylinder.

The equation $x^2=k$ represents the two parallel planes $x=\pm\sqrt{k}$.

Ex. 1. The sum of the squares of the reciprocals of any three diameters of an ellipsoid which are mutually at right angles is constant.

If r_1 be the semi-diameter whose direction-cosines are (l_1, m_1, n_1) we have $\frac{1}{r_1^2}=\frac{l_1^2}{a^2}+\frac{m_1^2}{b^2}+\frac{n_1^2}{c^2}$, and similarly for the other diameters. By addition we have $\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$.

Ex. 2. If three fixed points of a straight line are on given planes which are at right angles to one another, shew that any other point in the line describes an ellipsoid.

Let A, B, C be the points which are on the co-ordinate planes, and $P(x, y, z)$ be any other fixed point whose distances from A, B, C are a, b, c. Then $\frac{x}{a}=l$, $\frac{y}{b}=m$, and $\frac{z}{c}=n$, where l, m, n are the direction cosines of the line. Hence the equation of the locus is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Ex. 3. Find the equation of the cone whose vertex is at the centre of an ellipsoid and which passes through all the points of intersection of the ellipsoid and a given plane.

Let the equations of the ellipsoid and of the plane be $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, and $lx+my+nz=1$. We have only to make the equation of the ellipsoid homogeneous by means of the equation of the plane: the result is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=(lx+my+nz)^2.$$

For this equation being homogeneous represents a cone whose vertex is at the origin; and it is clear that the plane cuts the cone and the ellipsoid in the same points.

Ex. 4. Find the general equation of a cone of the second degree referred to three of its generators as axes of co-ordinates.

The general equation of a quadric cone whose centre is at the origin is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0.$$

If the axis of x be a generating line, then $y=0$, $z=0$ must satisfy the equation for all values of x; this gives $a=0$. Similarly, if the axes of y and z be generating lines, $b=0$ and $c=0$. Hence the most general form of the equation of a quadric cone referred to three generators as axes is

$$fyz+gzx+hxy=0.$$

Ex. 5. Find the equation of the cone whose vertex is at the centre of a given ellipsoid, and which goes through all points common to the ellipsoid and a concentric sphere.

If the equations of the ellipsoid and sphere be $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, and $x^2+y^2+z^2=r^2$ respectively; the equation of the cone will be

$$x^2\left(\frac{1}{a^2}-\frac{1}{r^2}\right)+y^2\left(\frac{1}{b^2}-\frac{1}{r^2}\right)+z^2\left(\frac{1}{c^2}-\frac{1}{r^2}\right)=0.$$

Ex. 6. Find the equation of the cone whose vertex is the point (α, β, γ) and whose generating lines pass through the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $z=0$.

Let any generator be $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$. This meets $z=0$ where $x=\alpha-\frac{l}{n}\gamma$, and $y=\beta-\frac{m}{n}\gamma$. Hence $\frac{1}{a^2}\left(\alpha-\frac{l}{n}\gamma\right)^2+\frac{1}{b^2}\left(\beta-\frac{m}{n}\gamma\right)^2=1$, or $\frac{1}{a^2}(\alpha n-\gamma l)^2+\frac{1}{b^2}(\beta n-\gamma m)^2=n^2$. Substitute for l, m, n from the equations of the line, and we have $\frac{1}{a^2}(\alpha z-\gamma x)^2+\frac{1}{b^2}(\beta z-\gamma y)^2=(z-\gamma)^2$, the required equation.

74. If the origin be the *centre* of the surface, it is the middle point of all chords passing through it; hence if (x_1, y_1, z_1) be any point on the surface, the point $(-x_1, -y_1, -z_1)$ will also be on the surface.

Hence we have

$$ax_1^2+by_1^2+cz_1^2+2fy_1z_1+2gz_1x_1+2hx_1y_1+2ux_1+2vy_1 \\ +2wz_1+d=0,$$

and

$$ax_1^2+by_1^2+cz_1^2+2fy_1z_1+2gz_1x_1+2hx_1y_1-2ux_1-2vy_1 \\ -2wz_1+d=0;$$

therefore $$ux_1+vy_1+wz_1=0.$$

Since this equation holds for all points (x_1, y_1, z_1) on the surface, we must have u, v, w all zero.

Hence, when the origin is the centre of a conicoid, the coefficients of x, y and z are all zero.

75. *To find the co-ordinates of the centre of a conicoid.*

Let (ξ, η, ζ) be the centre of the surface; then if we take (ξ, η, ζ) for origin, the coefficients of x, y, and z in the transformed equation will all be zero. The transformed equation will be [Art. 44]

$$a(x+\xi)^2+b(y+\eta)^2+c(z+\zeta)^2+2f(y+\eta)(z+\zeta) \\ +2g(z+\zeta)(x+\xi)+2h(x+\xi)(y+\eta)+2u(x+\xi)+2v(y+\eta) \\ +2w(z+\zeta)+d=0.$$

Hence the equations giving the centre are

$$\left.\begin{aligned} a\xi+h\eta+g\zeta+u&=0,\\ h\xi+b\eta+f\zeta+v&=0,\\ \text{and}\qquad g\xi+f\eta+c\zeta+w&=0,\end{aligned}\right\}\ldots\ldots\ldots\ldots\text{(i)}.$$

Therefore

$$\frac{\xi}{\begin{vmatrix} h, & g, & u \\ b, & f, & v \\ f, & c, & w \end{vmatrix}} = \frac{-\eta}{\begin{vmatrix} a, & g, & u \\ h, & f, & v \\ g, & c, & w \end{vmatrix}} = \frac{\zeta}{\begin{vmatrix} a, & h, & u \\ h, & b, & v \\ g, & f, & w \end{vmatrix}}$$

$$= \frac{-1}{\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}}.$$

The equation of the conicoid when referred to the centre (ξ, η, ζ) as origin is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+d'=0\ldots\ldots\text{(ii)},$$

where $d' \equiv F(\xi, \eta, \zeta)$.
Multiply equations (i) in order by ξ, η, ζ and subtract the sum from $F(\xi, \eta, \zeta)$; then we have

$$d'=u\xi+v\eta+w\zeta+d\ldots\ldots\ldots\ldots\ldots\text{(iii)}.$$

From (i) and (iii) we have

$$\begin{vmatrix} a, & h, & g, & u \\ h, & b, & f, & v \\ g, & f, & c, & w \\ u, & v, & w, & d-d' \end{vmatrix}=0;$$

therefore $$d'\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} = \begin{vmatrix} a, & h, & g, & u \\ h, & b, & f, & v \\ g, & f, & c, & w \\ u, & v, & w, & d \end{vmatrix}\ldots\ldots\text{(iv)}.$$

The determinant on the right side of (iv) is called the *discriminant* of the function $F(x, y, z)$, and is denoted by the symbol Δ.

The determinant on the left side is the discriminant of the terms in $F(x, y, z)$ which are of the second degree; it is also the minor of d in the determinant Δ, and, as in Art. 57, we shall denote it by D. Equation (iv) may therefore be written

$$d'D = \Delta \quad \text{.........................(v).}$$

76. The equations for finding the centre can also be obtained from Art. 58 (i); for (ξ, η, ζ) will be the middle point of every chord which passes through (ξ, η, ζ), provided

$$\frac{dF}{d\xi} = \frac{dF}{d\eta} = \frac{dF}{d\zeta} = 0.$$

It should be noticed that the co-ordinates of the centre are given by the equations

$$\frac{\xi}{U} = \frac{\eta}{V} = \frac{\zeta}{W} = \frac{1}{D},$$

where U, V, W, D have the same meanings as in Art. 57.

77. If, by a change of rectangular axes through the same origin, $\quad ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$

becomes changed into

$$a'x^2 + b'y^2 + c'z^2 + 2f'yz + 2g'zx + 2h'xy;$$

then, since $x^2 + y^2 + z^2$ is unaltered by the change of axes,

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy - \lambda(x^2 + y^2 + z^2) \ldots \text{(i)}$$

will be changed into

$$a'x^2 + b'y^2 + c'z^2 + 2f'yz + 2g'zx + 2h'xy \\ - \lambda(x^2 + y^2 + z^2) \ldots\ldots \text{(ii)}.$$

The expressions (i) and (ii) will therefore be the product of linear factors for the same values of λ.

The condition that (i) is the product of linear factors is

$$\begin{vmatrix} a-\lambda, & h, & g \\ h, & b-\lambda, & f \\ g, & f, & c-\lambda \end{vmatrix} = 0,$$

that is

$$\lambda^3 - \lambda^2(a+b+c) + \lambda(bc+ca+ab-f^2-g^2-h^2) \\ -(abc+2fgh-af^2-bg^2-ch^2)=0.$$

The condition that (ii) is the product of linear factors is similarly

$$\lambda^3 - \lambda^2(a'+b'+c') + \lambda(b'c'+c'a'+a'b'-f'^2-g'^2-h'^2) \\ -(a'b'c'+2f'g'h'-a'f'^2-b'g'^2-c'h'^2)=0.$$

Since the roots of the above cubic equations in λ are the same, the coefficients must be equal.

Hence the following expressions are unaltered by any change of rectangular axes through the same origin, and are therefore called *invariants:*

$$a+b+c \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{I},$$
$$bc+ca+ab-f^2-g^2-h^2 \ldots\ldots\ldots \text{II},$$
$$abc+2fgh-af^2-bg^2-ch^2 \ \ldots\ldots \text{III}.$$

Since the coefficients of the terms of the second degree are unaltered by a change of origin, the axes being parallel to their original directions, it follows that the expressions I, II, and III are unaltered by any change of rectangular axes.

78. We have seen [Art. 63] that by a proper choice of rectangular axes $ax^2+by^2+cz^2+2fyz+2gzx+2hxy$ can always be reduced to the form $\alpha x^2+\beta y^2+\gamma z^2$; and this reduction can be effected without changing the origin, for the terms of the second degree are not altered by transforming to any parallel axes.

Now $x^2+y^2+z^2$ is unaltered by a change of rectangular axes through the same origin. Hence, when the axes are so changed that

$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$ becomes $\alpha x^2 + \beta y^2 + \gamma z^2$,

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy - \lambda (x^2 + y^2 + z^2) \quad \text{...(i)},$$

will become

$$\alpha x^2 + \beta y^2 + \gamma z^2 - \lambda (x^2 + y^2 + z^2) \ldots\ldots\ldots\ldots\ldots \text{(ii)}.$$

Both these expressions will therefore be the product of linear factors for the same values of λ. The condition that (i) is the product of linear factors is

$$\begin{vmatrix} a-\lambda, & h\ , & g \\ h\ , & b-\lambda, & f \\ g\ , & f\ , & c-\lambda \end{vmatrix} = 0 \ldots\ldots\ldots\ldots\ldots \text{(iii)}.$$

But (ii) is the product of linear factors when λ is equal to α, β, or γ.

Hence the coefficients α, β, γ are the three roots of the equation (iii).

The equation when expanded is

$$\lambda^3 - \lambda^2 (a + b + c) + \lambda (ab + bc + ca - f^2 - g^2 - h^2) \\ - (abc + 2fgh - af^2 - bg^2 - ch^2) = 0.$$

This equation is called the *discriminating cubic.*

It should be noticed that the equation is the same as that found in Art. 60.

79. We proceed to shew how to find the nature of a conicoid whose equation is given.

First write down the equations for finding the centre of the conicoid; and from Art. 75 we see that there is a definite centre at a finite distance, unless the determinant

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} \equiv D$$

is zero.

If D be not zero, change to parallel axes through the centre, and the equation becomes

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + d' = 0,$$

where d' is found as in Art. 75.

Now, keeping the origin fixed, change the axes in such a manner that the equation is reduced to the form

$$\alpha x^2 + \beta y^2 + \gamma z^2 + d' = 0.$$

Then, by Art. 78, α, β, γ will be the three roots of the discriminating cubic.

[When the discriminating cubic cannot be solved, since its roots are all real [Art. 64], the number of positive and of negative roots can be found by Descartes' Rule of Signs.]

Since $Dd' = \Delta$, the last equation may be written in the form
$$D\alpha x^2 + D\beta y^2 + D\gamma z^2 + \Delta = 0.$$

If the three quantities $\dfrac{D\alpha}{\Delta}$, $\dfrac{D\beta}{\Delta}$, $\dfrac{D\gamma}{\Delta}$ are all negative, the surface is an *ellipsoid*; if two of them are negative, the surface is an *hyperboloid of one sheet*; if one is negative, the surface is an *hyperboloid of two sheets*; and if they are all positive, the surface is an *imaginary ellipsoid*.

If $\Delta = 0$, the surface is a *cone*.

Ex. (i). $11x^2 + 10y^2 + 6z^2 - 8yz + 4zx - 12xy + 72x - 72y + 36z + 150 = 0.$

The equations for finding the centre are $\dfrac{dF}{dx} = \dfrac{dF}{dy} = \dfrac{dF}{dz} = 0$, or

$$\begin{aligned} 11x - 6y + 2z + 36 &= 0, \\ -6x + 10y - 4z - 36 &= 0, \\ 2x - 4y + 6z + 18 &= 0. \end{aligned}$$

Therefore the centre is $(-2, 2, -1)$.

The equation referred to parallel axes through the centre will therefore be

$$11x^2 + 10y^2 + 6z^2 - 8yz + 4zx - 12xy - 12 = 0. \text{ [Art. 75 (iii).]}$$

The Discriminating Cubic is $\lambda^3 - 27\lambda^2 + 180\lambda - 324 = 0$; the roots of which are 3, 6, 18. Hence the equation represents the ellipsoid $3x^2 + 6y^2 + 18z^2 = 12$,

or
$$\frac{x^2}{4} + \frac{y^2}{2} + \frac{z^2}{\frac{2}{3}} = 1.$$

We can find the equations of the axes by using the formulae found in Art. 60. The direction-cosines of the axes are $\frac{1}{3}$, $\frac{2}{3}$, $\frac{2}{3}$; $\frac{2}{3}$, $\frac{1}{3}$, $-\frac{2}{3}$; $-\frac{2}{3}$, $\frac{2}{3}$, $-\frac{1}{3}$.

why not $6x^2 + 3y^2$

Ex. (ii). $x^2+2y^2+3z^2-4xz-4xy+d=0$.

The Discriminating Cubic is $\lambda^3-6\lambda^2+3\lambda+14=0$. All the roots of the cubic are real; hence, by Descartes' Rule of Signs, there are two positive roots and one negative root. The surface is therefore an hyperboloid of one sheet, an hyperboloid of two sheets, or a cone, according as d is negative, positive, or zero.

80. Next suppose that $D=0$. Then the three planes [Art. 75 (i)] on which the centre lies will not intersect in a point at a finite distance from the origin, and we shall have three cases to consider according as the planes meet in a point at infinity, or have a common line of intersection, or are all parallel to one another. These three cases we shall consider in the following Articles.

It should be observed that when $D=0$ one root of the discriminating cubic is zero.

81. The conditions that the planes whose equations are

$$ax+hy+gz+u=0,$$

$$hx+by+fz+v=0,$$

and

$$gx+fy+cz+w=0,$$

may be parallel are

$$\frac{a}{h}=\frac{h}{b}=\frac{g}{f} \text{ and } \frac{h}{g}=\frac{b}{f}=\frac{f}{c}.$$

These conditions may be written

$$af=gh,\ bg=hf,\ ch=fg \text{..................(i)}.$$

Now these are the conditions that the terms of the second degree should be a perfect square; and when this is the case it is obvious on inspection.

When the terms of the second degree are a perfect square, the general equation can be written in the form

$$fgh\left(\frac{x}{f}+\frac{y}{g}+\frac{z}{h}\right)^2+2ux+2vy+2wz+d=0 \text{.........(ii)}.$$

If the plane $ux+vy+wz=0$ is parallel to the plane

$$\frac{x}{f}+\frac{y}{g}+\frac{z}{h}=0,$$

the equation (ii) will represent *two parallel planes:* the conditions for this are

$$uf = vg = wh \text{ ....(iii).}$$

If the conditions (iii) are not satisfied, the equation (ii) is of the form $$Ay^2 + Bx = 0,$$

which represents a *parabolic cylinder* whose generating lines are parallel to $y = 0$, $x = 0$.

Hence the general equation of the second degree represents a *parabolic cylinder* whose generating lines are parallel to the line

$$\frac{x}{f} + \frac{y}{g} + \frac{z}{h} = 0, \quad ux + vy + wz = 0,$$

provided the conditions (i) are satisfied, and that (iii) are not satisfied.

The latus-rectum of the principal parabolic section can be found by the same method as that employed in Conics, Art. 172.

Ex. Find the nature of the conicoid whose equation is

$$4x^2 + y^2 + 4z^2 - 4yz + 8zx - 4xy + 2x - 4y + 5z + 1 = 0.$$

The equation is

$$(2x - y + 2z)^2 + 2x - 4y + 5z + 1 = 0.$$

This is equivalent to

$$(2x - y + 2z + \lambda)^2 = x(4\lambda - 2) - y(2\lambda - 4) + z(4\lambda - 5) - 1.$$

The planes $2x - y + 2z + \lambda = 0$, and $x(4\lambda - 2) - y(2\lambda - 4) + z(4\lambda - 5) - 1 = 0$, will be perpendicular, if $\lambda = 1$. Hence the equation of the surface may be written $$(2x - y + 2z + 1)^2 = 2x + 2y - z - 1,$$

or $$\left(\frac{2x - y + 2z + 1}{3}\right)^2 = \frac{1}{3} \cdot \frac{2x + 2y - z - 1}{3}.$$

Hence, taking $2x - y + 2z + 1 = 0$, and $2x + 2y - z - 1 = 0$ as the planes $y = 0$ and $x = 0$ respectively, the equation of the surface will be

$$y^2 = \frac{1}{3} x.$$

Hence the latus-rectum of a principal parabolic section is $\frac{1}{3}$.

82. Next suppose that the three planes on which the centre lies are not all parallel, but that they have a common line of intersection.

If we take any point on the line of centres for origin, the equation will take the form

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + d' = 0.$$

Then, keeping the origin fixed, by transformation of axes the equation will be reduced to the form

$$\alpha x^2 + \beta y^2 + d' = 0 \text{..........................(i)}.$$

One root of the discriminating cubic is zero, since $D = 0$; and the roots α, β, 0 are given by the equation

$$\lambda^3 - \lambda^2 (a + b + c) + \lambda (bc + ca + ab - f^2 - g^2 - h^2) = 0.$$

If $d' = 0$, the surface represented by the equation (i) is *two planes*, real or imaginary.

If d' be not zero, the surface is a cylinder.

The conditions that the three planes

$$ax + hy + gz + u = 0,$$
$$hx + by + fz + v = 0,$$
$$gx + fy + cz + w = 0,$$

may have a common line of intersection, are given by

$$\left\|\begin{matrix} a, & h, & g, & u \\ h, & b, & f, & v \\ g, & f, & c, & w \end{matrix}\right\| = 0, \text{ [Art. 18]}$$

that is, $$U = V = W = D = 0.$$

Ex. Find the nature of the conicoid whose equation is

$$32x^2 + y^2 + 4z^2 - 16zx - 8xy + 96x - 20y - 8z + 103 = 0.$$

The equations giving the centre are

$$32x - 4y - 8z + 48 = 0,$$
$$-4x + y \quad - 10 = 0,$$

and $$-8x \quad + 4z - 4 = 0.$$

Hence there is a line of centres. Find one point on the line, for example (0, 10, 1), and change the origin to the point (0, 10, 1): the equation will then become $$32x^2 + y^2 + 4z^2 - 16zx - 8xy = 1.$$

The Discriminating Cubic is $\lambda^3 - 37\lambda^2 + 84\lambda = 0$. One root is zero, and the other two roots are positive; hence the equation is an elliptic cylinder.

The axis of the cylinder is the line of centres; and its equations are

$$\frac{x}{1} = \frac{y-10}{4} = \frac{z-1}{2}.$$

83. If the planes on which the centre lies meet at a point at infinity, we proceed as follows.

Since one root of the discriminating cubic is zero, the equation can always be solved: let the roots be α, β, 0.

Find the directions of the principal axes of the surface, by means of the equations of Art. 60; and take axes parallel to these principal axes. The equation will then become

$$\alpha x^2 + \beta y^2 + 2u'x + 2v'y + 2w'z + d = 0,$$

or, by a change of origin,

$$\alpha x^2 + \beta y^2 + 2w'z = 0.$$

Hence the surface is a paraboloid, the latera recta of its principal parabolic sections being $\frac{2w'}{\alpha}$ and $\frac{2w'}{\beta}$.

Ex. Find the nature of the surface whose equation is

$$3z^2 - 6yz - 6zx - 7x - 5y + 6z + 3 = 0.$$

The Discriminating Cubic is $\lambda^3 - 3\lambda^2 - 18\lambda = 0$; the roots of which are 6, -3, 0.

The direction-cosines of the principal axes are $\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}$; $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$; and $\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0$. Hence to find the equation referred to axes parallel to the principal axes, we must substitute

$$\frac{x}{\sqrt{6}} + \frac{y}{\sqrt{3}} + \frac{z}{\sqrt{2}}, \quad \frac{x}{\sqrt{6}} + \frac{y}{\sqrt{3}} - \frac{z}{\sqrt{2}} \quad \frac{-2x}{\sqrt{3}} + \frac{y}{\sqrt{3}}$$

for x, y, z respectively. The equation will then become

$$6x^2 - 3y^2 - 4\sqrt{6}x - 2\sqrt{3}y - \sqrt{2}z + 3 = 0;$$

or, by changing the origin $\quad 6x^2 - 3y^2 - \sqrt{2}z = 0.$

Thus the surface is a hyperbolic paraboloid, the latera recta of the principal parabolas being $\frac{1}{6}\sqrt{2}$ and $\frac{1}{3}\sqrt{2}$.

84. It follows from Art. 75 (ii) and (iv) that when D is not zero, the necessary and sufficient condition that the surface represented by the general equation of the second degree may be a cone is $\Delta = 0$.

When $\Delta = 0$ and also $D = 0$, then will U, V and W be all zero*: hence [Arts. 81 and 82] the surface must be either a cylinder or two planes; and cylinders and planes are limiting forms of cones. Conversely, when the surface represents a cylinder, or two planes, U, V, W and D are all zero, and therefore also $\Delta = 0$.

Hence $\Delta = 0$ is the necessary and sufficient condition that the surface represented by the general equation of the second degree may be a cone.

85. *To find the conditions that the surface represented by the general equation of the second degree may be a surface of revolution.*

We require the condition that two of the roots of the discriminating cubic may be equal. In that case

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$

can be transformed into

$$\alpha x^2 + \alpha y^2 + \gamma z^2.$$

Hence

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy - \lambda(x^2 + y^2 + z^2)\ldots\text{(i)},$$

* This can be proved as follows:

We have $$uU + vV + wW + dD = \Delta.$$

And, since a determinant vanishes when two of its rows are identical, we have also

$$aU + hV + gW + uD = 0,$$
$$hU + bV + fW + vD = 0,$$
and
$$gU + fV + cW + wD = 0.$$

Hence when $\Delta = 0$ and $D = 0$, unless U, V, W are all zero, we can eliminate U, V, W from the first equation and any two of the others: we thus obtain three determinants which are all zero; but these determinants are U, V, and W.

can be transformed into

$$\alpha x^2 + \alpha y^2 + \gamma z^2 - \lambda (x^2 + y^2 + z^2) \ldots\ldots\ldots \text{(ii)}.$$

Now if we take $\lambda = \alpha$, (ii) will be a perfect square.

Hence if the surface is a surface of revolution, we can, by a proper choice of λ, make (i) a perfect square; and that square must be

$$\{x\sqrt{(a-\lambda)} + y\sqrt{(b-\lambda)} + z\sqrt{(c-\lambda)}\}^2.$$

We therefore have

$$\left.\begin{aligned} \sqrt{(b-\lambda)}\sqrt{(c-\lambda)} &= f \\ \sqrt{(c-\lambda)}\sqrt{(a-\lambda)} &= h \\ \sqrt{(a-\lambda)}\sqrt{(b-\lambda)} &= g \end{aligned}\right\} \ldots\ldots\ldots\ldots \text{(iii)}.$$

Hence, *if f, g, h be all finite,* we have

$$a - \frac{gh}{f} = b - \frac{hf}{g} = c - \frac{fg}{h} = \lambda \ldots\ldots\ldots\ldots \text{(iv)},$$

the required conditions.

Let h, any one of the three quantities f, g, h, be zero; then from (iii) we see that $\lambda = a$ or $\lambda = b$, and therefore also $g = 0$ or $f = 0$.

Suppose $g = 0$ and $h = 0$; then $\lambda = a$, and the condition for a surface of revolution is

$$(b-a)(c-a) = f^2 \ldots\ldots\ldots\ldots\ldots\ldots \text{(v)}.$$

Examples on Chapter III.

1. Determine the nature of the surfaces represented by the following equations:

(i) $x^2 - 2y^2 + 6z^2 + 12xz + a^2 = 0$.

2,2,-1,-16 (ii) $x^2 + y^2 + z^2 + 4xy - 2xz + 4yz = 1$.

(iii) $x^2 - 2xy - 2yz - 2zx = a^2$.

(iv) $32x^2 + y^2 + 4z^2 - 16zx - 8xy = 1$.

(v) $\sqrt{x} + \sqrt{y} + \sqrt{z} = 0$.

(vi) $2x^2 + 5y^2 + z^2 - 4xy - 2x - 4y - 8 = 0$.

2. Find the nature of the surfaces represented by the following equations:

(i) $x^2 + 2y^2 - 3z^2 - 4yz + 8zx - 12xy + 1 = 0$.

(ii) $2x^2 + 2y^2 - 4z^2 - 2yz - 2zx - 5xy - 2x - 2y + z = 0$.

(iii) $5x^2 - y^2 + z^2 + 6xz + 4xy + 2x + 4y + 6z = 8$.

(iv) $2x^2 + 3y^2 + 3yz + 2zx + 5xy - 4y + 8z - 32 = 0$.

Find the equations of the axes of (i), and the latera recta of the principal parabolas of (ii) and of (iii).

3. Shew that the equation

$$x^2 + y^2 + z^2 + yz + zx + xy = 1,$$

represents an ellipsoid the squares of whose semi-axes are 2, 2, $\frac{1}{2}$. Shew also that the equation of its principal axis is $x = y = z$.

4. Shew that, if the axes, supposed rectangular, be turned round the origin in any manner, $u^2 + v^2 + w^2$ will be unaltered.

5. Shew that, if three chords of a conicoid have the same middle point, they all lie in a plane, or intersect in the centre of the conicoid.

6. Through any point O lines are drawn in fixed directions which meet a given conicoid in points P, P' and Q, Q' respectively; shew that the rectangles OP, OP' and OQ, OQ' are in a constant ratio.

7. If any three rectangular axes through a fixed point O cut a given conicoid in P, P'; Q, Q' and R, R'; then will

$$\frac{PP'^2}{OP^2 . OP'^2} + \frac{QQ'^2}{OQ^2 . OQ'^2} + \frac{RR'^2}{OR^2 . OR'^2},$$

and

$$\frac{1}{OP . OP'} + \frac{1}{OQ . OQ'} + \frac{1}{OR . OR'},$$

be constant.

CHAPTER IV.

CONICOIDS REFERRED TO THEIR AXES.

86. IN the present chapter we shall investigate some properties of conicoids, obtained by taking the equations of the surfaces in the simplest forms to which they can be reduced.

We shall begin by considering the *Sphere*.

THE SPHERE.

87. The equation of the sphere whose centre is (a, b, c) and radius d is [Art. 5]

$$(x-a)^2+(y-b)^2+(z-c)^2=d^2.$$

The equation of any sphere is therefore of the form

$$x^2+y^2+z^2+2Ax+2By+2Cz+D=0.$$

Conversely every equation of the above form, that is every equation in which the coefficients of x^2, y^2, and z^2 are equal, and in which the terms yz, zx, xy do not appear, represents a sphere.

88. The general equation of a sphere contains four constants, and therefore a sphere can be made to satisfy *four* conditions. We may, for example, find the equation of a sphere which passes through any four points.

If (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4) be the four points the equation of the sphere through them will be.

$$\begin{vmatrix} x^2+y^2+z^2, & x, & y, & z, & 1 \\ x_1^2+y_1^2+z_1^2, & x_1, & y_1, & z_1, & 1 \\ x_2^2+y_2^2+z_2^2, & x_2, & y_2, & z_2, & 1 \\ x_3^2+y_3^2+z_3^2, & x_3, & y_3, & z_3, & 1 \\ x_4^2+y_4^2+z_4^2, & x_4, & y_4, & z_4, & 1 \end{vmatrix} = 0.$$

89. The equation of the tangent plane at any point (x', y', z') of the sphere whose equation is $x^2+y^2+z^2=a^2$ is $xx'+yy'+zz'=a^2$ [Art. 52, Ex. 1]. This result can be obtained at once from the fact that the tangent plane at any point (x', y', z') on a sphere is perpendicular to the line joining (x', y', z') to the centre. This gives for the equation of the plane

$$(x-x')x'+(y-y')y'+(z-z')z'=0,$$

or
$$xx'+yy'+zz'=a^2.$$

The polar plane of any point (x', y', z') can be shewn, by the method of Art. 53, to be

$$xx'+yy'+zz'=a^2.$$

90. It can be easily shewn, that if $S=0$ be the equation of a sphere (where S is written for shortness instead of $x^2+y^2+z^2+2Ax+2By+2Cz+D$), and the co-ordinates of any point be substituted in S, the result will be equal to the square of the tangent from that point to the sphere.

Hence, if $S=0$, and $S'=0$ be the equations of two spheres (in each of which the coefficient of x^2 is unity), $S=S'$ is the locus of points, the tangents from which to the two spheres are equal.

The surface whose equation is $S-S'=0$ passes through all points common to the two spheres $S=0$, and $S'=0$; for, if the co-ordinates of any point satisfy the equations $S=0$ and $S'=0$, they will also satisfy the equation $S-S'=0$.

Now $S-S'=0$ is of the first degree, and therefore represents a *plane*. The plane through the points of intersection of two spheres is called their *radical plane*.

We have seen that the tangents drawn to two spheres from any point on their radical plane are equal.

The radical planes of four given spheres meet in a point, viz. in the point given by $S_1 = S_2 = S_3 = S_4$, where $S_1 = 0$, $S_2 = 0$, $S_3 = 0$, $S_4 = 0$ are the equations of the four spheres, in each of which the coefficient of x^2 is unity.

This point is called the *radical centre* of the four spheres.

Ex. 1. Find the equation of the sphere which has (x_1, y_1, z_1) and (x_2, y_2, z_2) for extremities of a diameter.

If (x, y, z) be any point on the sphere, the direction-cosines of the lines joining (x, y, z) to the two given points are proportional to $x - x_1$, $y - y_1$, $z - z_1$, and $x - x_2$, $y - y_2$, $z - z_2$.

The condition of perpendicularity of these lines gives the required equation

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0.$$

Ex. 2. The locus of a point, the sum of the squares of whose distances from any number of given points is constant, is a sphere.

Ex. 3. A point moves so that the sum of the squares of its distances from the six faces of a cube is constant; shew that its locus is a sphere.

Ex. 4. A, B are two fixed points, and P moves so that $PA = nPB$; shew that the locus of P is a sphere. Shew also that all such spheres, for different values of n, have a common radical plane.

Ex. 5. The distances of two points from the centre of a sphere are proportional to the distance of each from the polar of the other.

Ex. 6. Shew that the spheres whose equations are

$$x^2 + y^2 + z^2 + 2Ax + 2By + 2Cz + D = 0,$$

and

$$x^2 + y^2 + z^2 + 2ax + 2by + 2cz + d = 0,$$

cut one another at right angles, if

$$2Aa + 2Bb + 2Cc - D - d = 0.$$

91. We proceed to prove some properties of the ellipsoid; and we shall always suppose the equation of the surface to be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

unless it is otherwise expressed.

To obtain the properties of the hyperboloids we shall only have to make the necessary changes in the signs of b^2 and c^2.

We have already seen [Art. 52] that the equation of the tangent plane at any point (x', y', z') is

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}+\frac{zz'}{c^2}=1 \qquad\text{(i)}.$$

The length of the perpendicular from the origin on the tangent plane at the point (x', y', z') is [Art. 20] given by the equation

$$\frac{1}{p^2}=\frac{x'^2}{a^4}+\frac{y'^2}{b^4}+\frac{z'^2}{c^4} \qquad\text{(ii)}.$$

Equation (i) is equivalent to $lx+my+nz=p$, where

$$\frac{l}{p}=\frac{x'}{a^2},\quad \frac{m}{p}=\frac{y'}{b^2},\quad \frac{n}{p}=\frac{z'}{c^2};$$

therefore $$\frac{a^2l^2+b^2m^2+c^2n^2}{p^2}=\frac{x'^2}{a^2}+\frac{y'^2}{b^2}+\frac{z'^2}{c^2}=1.$$

Hence the plane whose equation is $lx+my+nz=p$, will touch the ellipsoid, if

$$p^2=a^2l^2+b^2m^2+c^2n^2 \qquad\text{(iii)}.$$

92. *To find the locus of the point of intersection of three tangent planes to an ellipsoid which are mutually at right angles.*

Let the equations of the planes be

$$l_1x+m_1y+n_1z=\sqrt{(a^2l_1^2+b^2m_1^2+c^2n_1^2)},$$
$$l_2x+m_2y+n_2z=\sqrt{(a^2l_2^2+b^2m_2^2+c^2n_2^2)},$$
$$l_3x+m_3y+n_3z=\sqrt{(a^2l_3^2+b^2m_3^2+c^2n_3^2)}.$$

By squaring both sides of these equations and adding, we have in virtue of the relations between the direction-cosines of perpendicular lines

$$x^2+y^2+z^2=a^2+b^2+c^2.$$

The required locus is therefore a sphere. This sphere is called the *director-sphere* of the ellipsoid.

93. The normal to a surface at any point P is the straight line through P perpendicular to the tangent plane at P.

The normal to an ellipsoid at the point (x', y', z') is therefore

$$\frac{x-x'}{\frac{x'}{a^2}}=\frac{y-y'}{\frac{y'}{b^2}}=\frac{z-z'}{\frac{z'}{c^2}}.$$

Since $$p^2\left(\frac{x'^2}{a^4}+\frac{y'^2}{b^4}+\frac{z'^2}{c^4}\right)=1,$$ [Art. 91.]

the direction-cosines of the normal are

$$\frac{px'}{a^2},\ \frac{py'}{b^2},\ \frac{pz'}{c^2}.$$

94. If the normal at (x', y', z') pass through the particular point (f, g, h) we have

$$\frac{f-x'}{\frac{x'}{a^2}}=\frac{g-y'}{\frac{y'}{b^2}}=\frac{h-z'}{\frac{z'}{c^2}}.$$

Put each fraction equal to λ, then

$$x'=\frac{a^2f}{a^2+\lambda},\ y'=\frac{b^2g}{b^2+\lambda}\text{ and } z'=\frac{c^2h}{c^2+\lambda}.$$

Hence, since

$$\frac{x'^2}{a^2}+\frac{y'^2}{b^2}+\frac{z'^2}{c^2}=1,$$

we have

$$\frac{a^2f^2}{(a^2+\lambda)^2}+\frac{b^2g^2}{(b^2+\lambda)^2}+\frac{c^2h^2}{(c^2+\lambda)^2}=1.$$

Since this equation for λ is of the *sixth* degree, it follows that there are *six* points the normals at which pass through a given point.

Ex. 1. The normal at any point P of an ellipsoid meets a principal plane in G. Shew that the locus of the middle point of PG is an ellipsoid. ✓

Ex. 2. The normal at any point P of an ellipsoid meets the principal planes in G_1, G_2, G_3. Shew that PG_1, PG_2, PG_3 are in a constant ratio. ✓

Ex. 3. The normals to an ellipsoid at the points P, P' meet a principal plane in G, G'; shew that the plane which bisects PP' at right angles bisects GG'. ✓

Ex. 4. If P, Q be any two points on an ellipsoid, the plane through the centre and the line of intersection of the tangent planes at P, Q, will bisect PQ.

Ex. 5. P, Q are any two points on an ellipsoid, and planes through the centre parallel to the tangent planes at P, Q cut the chord PQ in P', Q'. Shew that $PP'=QQ'$.

95. The line whose equations are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r,$$

meets the surface where

$$\frac{(\alpha+lr)^2}{a^2}+\frac{(\beta+mr)^2}{b^2}+\frac{(\gamma+nr)^2}{c^2}=1.$$

If (α, β, γ) be the middle point of the chord, the two values of r given by the above equation must be equal and opposite; therefore the coefficient of r is zero, so that we have

$$\frac{l\alpha}{a^2}+\frac{m\beta}{b^2}+\frac{n\gamma}{c^2}=0.$$

Hence the middle points of all chords of the ellipsoid which are parallel to the line

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$$

are on the plane whose equation is

$$\frac{lx}{a^2}+\frac{my}{b^2}+\frac{nz}{c^2}=0.$$

This plane is called the *diametral plane* of the line

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}.$$

The diametral plane of lines parallel to the diameter through the point (x', y', z') on the surface is

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}+\frac{zz'}{c^2}=0\dots\dots\dots\dots\text{(i)};$$

hence the diametral plane of any diameter is parallel to the tangent plane at the extremities of that diameter.

The condition that the point (x'', y'', z'') should be on the diametral plane (i) is

$$\frac{x'x''}{a^2}+\frac{y'y''}{b^2}+\frac{z'z''}{c^2}=0.$$

The symmetry of this result shews that if a point Q be on the diametral plane of OP, then will P be on the diametral plane of OQ.

Let OR be the line of intersection of the diametral planes of OP, OQ; then, since the diametral planes of OP, OQ pass through OR, the diametral plane of OR will pass through P and through Q, and will therefore be the plane POQ, so that the plane through any two of the three lines OP, OQ, OR is diametral to the third.

Three planes are said to be *conjugate* when each is diametral to the line of intersection of the other two, and three diameters are said to be conjugate when the plane of any two is diametral to the third.

96. If (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) be extremities of conjugate diameters, we have from Art. 95,

$$\left.\begin{aligned}\frac{x_2x_3}{a^2}+\frac{y_2y_3}{b^2}+\frac{z_2z_3}{c^2}&=0\\ \frac{x_3x_1}{a^2}+\frac{y_3y_1}{b^2}+\frac{z_3z_1}{c^2}&=0\\ \frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}+\frac{z_1z_2}{c^2}&=0\end{aligned}\right\}\text{................(i).}$$

Also, since the points are on the surface,

$$\left.\begin{aligned}\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}&=1\\ \frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}+\frac{z_2^2}{c^2}&=1\\ \frac{x_3^2}{a^2}+\frac{y_3^2}{b^2}+\frac{z_3^2}{c^2}&=1\end{aligned}\right\}\text{................(ii).}$$

Now from equations (ii) we see that

$$\frac{x_1}{a}, \frac{y_1}{b}, \frac{z_1}{c}; \frac{x_2}{a}, \frac{y_2}{b}, \frac{z_2}{c}; \text{ and } \frac{x_3}{a}, \frac{y_3}{b}, \frac{z_3}{c};$$

are direction-cosines of three straight lines, and from equations (i) we see that the straight lines are two and two at right angles. Hence, as in Art. 45, we have

$$\left.\begin{aligned} x_1^2 + x_2^2 + x_3^2 = a^2 \\ y_1^2 + y_2^2 + y_3^2 = b^2 \\ z_1^2 + z_2^2 + z_3^2 = c^2 \end{aligned}\right\} \text{..................(iii)},$$

and

$$\left.\begin{aligned} x_1y_1 + x_2y_2 + x_3y_3 = 0 \\ y_1z_1 + y_2z_2 + y_3z_3 = 0 \\ z_1x_1 + z_2x_2 + z_3x_3 = 0 \end{aligned}\right\} \text{...............(iv)}.$$

We have also from Art. 46.

$$\begin{vmatrix} \frac{x_1}{a}, & \frac{y_1}{b}, & \frac{z_1}{c} \\ \frac{x_2}{a}, & \frac{y_2}{b}, & \frac{z_2}{c} \\ \frac{x_3}{a}, & \frac{y_3}{b}, & \frac{z_3}{c} \end{vmatrix} = 1, \text{ or } \begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix} = abc \text{ ....(v)}.$$

From (iii) we see that the sum of the squares of the projections of three conjugate semi-diameters of an ellipsoid on any one of its axes is constant.

Also, by addition, we have, *the sum of the squares of three conjugate diameters of an ellipsoid is constant.*

From (v) we see that *the volume of the parallelopiped which has three conjugate semi-diameters of an ellipsoid for conterminous edges is constant.*

In the above the relations (iii) and (iv) were deduced from (i) and (ii) by geometrical considerations. They could however be deduced by the ordinary processes of algebra without any consideration of the geometrical meaning of the quantities, and hence the results are true for the hyperboloids.

97. The two propositions (1) that the sum of the squares of three conjugate semi-diameters is constant, and (2) that the parallelopiped which has three conjugate semi-diameters for conterminous edges is of constant volume, are extremely important. We append other proofs of these propositions.

Since in any conic the sum of the squares of two conjugate semi-diameters is constant, and also the parallelogram of which they are adjacent sides, it follows that in any conicoid no change is made either in the sum of the squares or in the volume of the parallelopiped, so long as we keep *one* of the three conjugate diameters fixed.

We have therefore only to shew that we can pass from any system of conjugate diameters to the principal axes of the surface by a series of changes in each of which we keep one of the conjugate diameters fixed.

This can be proved as follows:—let OP, OQ, OR, be any three conjugate semi-diameters, and let the plane QOR cut a principal plane in the line OQ', and let OR' be in the plane QOR conjugate to OQ'; then OP, OQ', OR' are three conjugate semi-diameters.

Again, let the plane POR' meet the principal plane in which OQ' lies in the line OP'', and let OR'' be conjugate to OP'' and in the plane POR'; then OP'', OQ' and OR'' are semi-conjugate diameters. But, since OR'' is conjugate to OP'' and to OQ', both of which are in a principal plane, it must be a principal diameter.

Hence, finally, we have only to take the axes of the section $Q'OP''$ to have the three principal diameters.

98. It is known that any two conjugate diameters of a conic will both meet the curve in real points when it is an ellipse; that *one* will meet the curve in imaginary points when it is an hyperbola; and that *both* will meet the curve in imaginary points when it is an imaginary ellipse. Hence, by transforming as in the preceding Article, we see that three conjugate diameters of a conicoid will all meet the surface in real points when it is an ellipsoid; that *one* will meet the surface in imaginary points when it is an hyper-

boloid of one sheet; and that *two* will meet the surface in imaginary points when it is an hyperboloid of two sheets.

99. *To find the equation of an ellipsoid referred to three conjugate diameters as axes.*

Since the origin is unaltered we substitute for x, y and z expressions of the form $lx + my + nz$ in order to obtain the transformed equation [Art. 47].

The equation of the ellipsoid will therefore be of the form

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 1.$$

By supposition the plane $x = 0$ bisects all chords parallel to the axis of x. Therefore if (x_1, y_1, z_1) be *any* point on the surface, $(-x_1, y_1, z_1)$ will also be on the surface. Hence $Gz_1x_1 + Hx_1y_1 = 0$ for *all* points on the surface: this requires that $G = H = 0$.

Similarly, since the plane $y = 0$ bisects all chords parallel to the axis of y, we have $H = F = 0$.

Hence the equation of the surface is

$$Ax^2 + By^2 + Cz^2 = 1,$$

or

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} + \frac{z^2}{c'^2} = 1,$$

where a', b', c' are the lengths of the semi-diameters.

100. We may obtain the relations between conjugate diameters of central conicoids by the following method:—The expression

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} + \lambda(x^2 + y^2 + z^2)$$

is transformed, by taking for axes three conjugate diameters which make angles α, β, γ with one another, into the expression

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} + \frac{z^2}{c'^2} + \lambda(x^2 + y^2 + z^2 + 2yz\cos\alpha + 2zx\cos\beta + 2xy\cos\gamma).$$

The two expressions will therefore both split up into linear factors for the same values of λ. Hence the roots of the cubics

$$\left(\frac{1}{a^2} + \lambda\right)\left(\frac{1}{b^2} + \lambda\right)\left(\frac{1}{c^2} + \lambda\right) = 0,$$

and
$$\begin{vmatrix} \frac{1}{a'^2}+\lambda, & \lambda\cos\gamma, & \lambda\cos\beta \\ \lambda\cos\gamma, & \frac{1}{b'^2}+\lambda, & \lambda\cos\alpha \\ \lambda\cos\beta, & \lambda\cos\alpha, & \frac{1}{c'^2}+\lambda \end{vmatrix} = 0$$

are equal to one another.

Hence, by comparing coefficients in the two equations, we have

$$a^2+b^2+c^2=a'^2+b'^2+c'^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)},$$

$$b^2c^2+c^2a^2+a^2b^2=b'^2c'^2\sin^2\alpha+c'^2a'^2\sin^2\beta+a'^2b'^2\sin^2\gamma \ldots\ldots\ldots \text{(ii)},$$

and

$$abc=a'b'c'\sqrt{(1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma)} \ldots \text{(iii)}.$$

Therefore the sum of the squares of three conjugate diameters is constant; the sum of the squares of the areas of the faces of a parallelopiped having three conjugate radii for conterminous edges is constant; and the volume of such a parallelopiped is constant.

Ex. 1. If a parallelopiped be inscribed in an ellipsoid, its edges will be parallel to conjugate diameters.

Ex. 2. Shew that the sum of the squares of the projections of three conjugate diameters of a conicoid on any line, or on any plane, is constant.

Ex. 3. The sum of the squares of the distances of a point from the six ends of any three conjugate diameters is constant; shew that the locus of the point is a sphere.

Ex. 4. If $(x_1y_1z_1)$, $(x_2y_2z_2)$, $(x_3y_3z_3)$ be extremities of three conjugate diameters of an ellipsoid, the equation of the plane through them will be

$$\frac{x}{a^2}(x_1+x_2+x_3)+\frac{y}{b^2}(y_1+y_2+y_3)+\frac{z}{c^2}(z_1+z_2+z_3)=1.$$

Ex. 5. Shew that the tangent planes at the extremities of three conjugate diameters of an ellipsoid meet on a similar ellipsoid.

Ex. 6. Shew that the locus of the centre of gravity of a triangle whose angular points are the extremities of three conjugate diameters of an ellipsoid is a similar ellipsoid.

The Paraboloids.

101. We have seen that the paraboloids are particular cases of the central surfaces; properties of the paraboloids can therefore be deduced from the corresponding properties of the central surfaces. We will, however, investigate some of the properties independently.

We shall always suppose the equation of the surface to be

$$\frac{x^2}{a} + \frac{y^2}{b} = 2z.$$

102. *To find the locus of the point of intersection of three tangent planes to a paraboloid which are mutually at right angles.*

Let $l_1x + m_1y + n_1z + p_1 = 0$ be one of the tangent planes; then, since the plane touches the surface, we have

$$al_1^2 + bm_1^2 = 2n_1 p_1. \qquad \text{[Art. 57, II.]}$$

Hence we may write the equation in the form

$$l_1n_1x + m_1n_1y + n_1^2 z + \tfrac{1}{2}(al_1^2 + bm_1^2) = 0.$$

We have also

$$l_2n_2 x + m_2n_2 y + n_2^2 z + \tfrac{1}{2}(al_2^2 + bm_2^2) = 0,$$

and

$$l_3n_3 x + m_3n_3 y + n_3^2 z + \tfrac{1}{2}(al_3^2 + bm_3^2) = 0.$$

Since the planes are at right angles, we have by addition

$$z + \tfrac{1}{2}(a + b) = 0;$$

hence the locus is a plane.

103. The equation of the normal at any point (x', y', z') of the paraboloid is

$$\frac{x - x'}{\dfrac{x'}{a}} = \frac{y - y'}{\dfrac{y'}{b}} = \frac{z - z'}{-1}.$$

The normal at (x', y', z') will pass through the particular point (f, g, h), if

$$\frac{f-x'}{\frac{x'}{a}} = \frac{g-y'}{\frac{y'}{b}} = \frac{h-z'}{-1}.$$

Put each fraction equal to λ; then

$$x' = \frac{af}{a+\lambda},\ y' = \frac{bg}{b+\lambda},\ z' = h+\lambda;$$

and substituting in

$$\frac{x'^2}{a} + \frac{y'^2}{b} = 2z',$$

we have

$$\frac{af^2}{(a+\lambda)^2} + \frac{bg^2}{(b+\lambda)^2} = 2(h+\lambda).$$

The equation in λ is of the fifth degree; therefore five normals can be drawn from any point to a paraboloid.

104. The middle points of all chords of the paraboloid which are parallel to the line

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$$

are [Art. 59] on the plane whose equation is

$$l\frac{dF}{dx} + m\frac{dF}{dy} + n\frac{dF}{dz} = 0,$$

or
$$\frac{lx}{a} + \frac{my}{b} - n = 0.$$

Hence all diametral planes are parallel to the axis of the surface.

It is easy to shew conversely that all planes parallel to the axis are diametral planes.

A line parallel to the axis of the surface is called a *diameter*. Every diameter meets the surface in *one* point at a finite distance from the origin; and this point is called the extremity of the diameter.

The two diametral planes whose equations are

$$\frac{lx}{a} + \frac{my}{b} - n = 0,$$

and
$$\frac{l'x}{a} + \frac{m'y}{b} - n' = 0,$$

are such that each is parallel to the chords bisected by the other, if

$$\frac{ll'}{a} + \frac{mm'}{b} = 0.$$

If this condition be satisfied, the planes are called conjugate diametral planes.

The condition shews that conjugate diametral planes meet the plane $z = 0$ in lines which are parallel to conjugate diameters of the conic

$$\frac{x^2}{a} + \frac{y^2}{b} = 1.$$

105. If we move the origin to any point (α, β, γ) on the surface, the equation becomes

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{2x\alpha}{a} + \frac{2y\beta}{b} - 2z = 0.$$

If we take the planes

$$x = 0,\ y = 0, \text{ and } \frac{x\alpha}{a} + \frac{y\beta}{b} - z = 0$$

as co-ordinate planes, and therefore the lines

$$\frac{x}{a} = \frac{y}{0} = \frac{z}{\alpha},\quad \frac{x}{0} = \frac{y}{b} = \frac{z}{\beta}, \text{ and } \frac{x}{0} = \frac{y}{0} = \frac{z}{1}$$

for axes, we must [Art. 47] substitute

$$\frac{ax}{\sqrt{(a^2+\alpha^2)}},\quad \frac{by}{\sqrt{(b^2+\beta^2)}},\quad \frac{\alpha x}{\sqrt{(a^2+\alpha^2)}} + \frac{\beta y}{\sqrt{(b^2+\beta^2)}} + z$$

for x, y, z respectively.

The transformed equation is

$$\frac{x^2}{a + \dfrac{\alpha^2}{a}} + \frac{y^2}{b + \dfrac{\beta^2}{b}} = 2z.$$

This is the equation to the surface referred to a point (α, β, γ) as origin, two of the co-ordinate planes being parallel to their original directions, and the third being the tangent plane at (α, β, γ).

Ex. 1. Shew that the locus of the centres of parallel sections of a paraboloid is a diameter. See p. 44

Ex. 2. Shew that all planes parallel to the axis of a paraboloid cut the surface in parabolas.

Ex. 3. Shew that the latera recta of all parallel parabolic sections of a paraboloid are equal.

Ex. 4. Shew that the projections, on a plane perpendicular to the axis of a paraboloid, of all plane sections which are not parallel to the axis, are similar conics.

Ex. 5. P, Q are any two points on a paraboloid, and the tangent planes at P, Q intersect in the line RS; shew that the plane through RS and the middle point of PQ is parallel to the axis of the paraboloid.

Ex. 6. Shew that two conjugate points on a diameter of a paraboloid are equidistant from the extremity of that diameter.

Ex. 7. Shew that the sum of the latera recta of the sections of a paraboloid, made by any two conjugate diametral planes through a fixed point on the surface, is constant.

CONES.

106. The general equation of a cone of the second degree is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0.$$

The tangent plane at any point (x', y', z') on the surface is

$$(x - x')(ax' + hy' + gz') + (y - y')(hx' + by' + fz') + (z - z')(gx' + fy' + cz') = 0,$$

or

$$x(ax' + hy' + gz') + y(hx' + by' + fz') + z(gx' + fy' + cz') = 0.$$

The form of this equation shews that the tangent plane at any point on a cone passes through its vertex, as is geometrically evident from the fact that the generating line through any point is one of the tangent lines at that point, and therefore lies in the tangent plane.

107. To find the condition that the plane $lx+my+nz=0$ may touch the cone whose equation is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0.$$

Comparing the equation of the tangent plane at the point (x', y', z'), namely

$$x(ax'+hy'+gz')+y(hx'+by'+fz')+z(gx'+fy'+cz')=0,$$

with the given equation, we have

$$\frac{ax'+hy'+gz'}{l}=\frac{hx'+by'+fz'}{m}=\frac{gx'+fy'+cz'}{n}.$$

Put each fraction equal to $-\lambda$, then

$$ax'+hy'+gz'+\lambda l=0,$$
$$hx'+by'+fz'+\lambda m=0,$$

and $$gx'+fy'+cz'+\lambda n=0.$$

Also, since (x', y', z') is on the plane,

$$lx'+my'+nz'=0.$$

Eliminating x', y', z', λ, we have the required condition

$$\begin{vmatrix} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c, & n \\ l, & m, & n, & 0 \end{vmatrix}=0,$$

or $$Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm=0,$$

where A, B, C, &c. are the minors of a, b, c, &c. in the determinant

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}.$$

108. If through the vertex of a given cone lines be drawn perpendicular to its tangent planes, these lines generate another cone called the *reciprocal* cone.

The line through the origin perpendicular to the plane

$$lx+my+nz=0, \text{ is } \frac{x}{l}=\frac{y}{m}=\frac{z}{n}.$$

Hence, from the result of the last article, the reciprocal of the cone

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$$

is $$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0.$$

Since the minors of A, B, C, &c. in the determinant

$$\begin{vmatrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{vmatrix}$$

are proportional to a, b, c, &c., we see that the relation between the two cones is a reciprocal one.

As a particular case of the above, the reciprocal of the cone

$$ax^2 + by^2 + cz^2 = 0, \text{ is } \frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0.$$

From this we see at once that *a cone and its reciprocal are co-axial.*

109. *To find the condition that a cone may have three perpendicular generators.*

Let the equation of the cone be

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \ldots\ldots \text{(i)}.$$

If the cone have three perpendicular generators, and we take these for axes of co-ordinates, the equation will [Art. 73, Ex. 4] take the form

$$Ayz + Bzx + Cxy = 0 \ \ldots\ldots\ldots\ldots\ldots \text{(ii)}.$$

Since the sum of the co-efficients of x^2, y^2 and z^2 is an invariant [Art. 79] and in (ii) the sum is zero; therefore the sum must be zero in (i) also. Therefore a *necessary* condition is

$$a + b + c = 0 \ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)}.$$

If the condition (iii) is satisfied there are an infinite number of sets of three perpendicular generators. For take *any* generator for the axis of x; then by supposition any point on the line $y = 0$, $z = 0$ is on the surface; therefore the co-efficient of x^2 is zero, so that the transformed equation is of the form

$$by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \quad \ldots\ldots\text{(iv)};$$

and since the sum of the co-efficients of x^2, y^2, z^2 is an invariant, we have $b + c = 0$.

Now the section of (iv) by the plane $x = 0$ is the two straight lines

$$by^2 + cz^2 + 2fyz = 0;$$

and these are at right angles, since $b + c = 0$.

110. If a cone have three perpendicular tangent planes, the reciprocal cone will have three perpendicular generators.

Hence the necessary and sufficient condition that the cone

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$$

may have three perpendicular tangent planes is

$$A + B + C = 0.$$

Ex. 1. CP, CQ, CR are three central radii of an ellipsoid which are mutually at right angles to one another; shew that the plane PQR touches a sphere.

Let the equation of the plane PQR be $lx + my + nz = p$. The equation of the cone whose vertex is the origin, and which passes through the intersection of the plane and the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \left(\frac{lx + my + nz}{p}\right)^2$. By supposition the cone has three perpendicular generators; therefore $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{p^2}$.

Ex. 2. Any two sets of rectangular axes which meet in a point form six generators of a cone of the second degree.

Ex. 3. Shew that any two sets of perpendicular planes which meet in a point all touch a cone of the second degree.

111. *To find the equation of the tangent cone from any point to an ellipsoid.*

Let the equation of the ellipsoid be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Let the co-ordinates of any two points P, Q be x', y', z' and x'', y'', z'' respectively.

The co-ordinates of a point which divides PQ in the ratio $m : n$ are

$$\frac{nx'+mx''}{m+n},\quad \frac{ny'+my''}{m+n},\quad \frac{nz'+mz''}{m+n}.$$

If this point be on the ellipsoid, we have

$$\frac{(nx'+mx'')^2}{a^2}+\frac{(ny'+my'')^2}{b^2}+\frac{(nz'+mz'')^2}{c^2}=(m+n)^2,$$

or
$$n^2\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}+\frac{z'^2}{c^2}-1\right)+2mn\left(\frac{x'x''}{a^2}+\frac{y'y''}{b^2}+\frac{z'z''}{c^2}-1\right)$$
$$+m^2\left(\frac{x''^2}{a^2}+\frac{y''^2}{b^2}+\frac{z''^2}{c^2}-1\right)=0.$$

If the line PQ cut the surface in coincident points, the above equation, considered as a quadratic in $\frac{n}{m}$, must have equal roots; the condition for this is

$$\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}+\frac{z'^2}{c^2}-1\right)\left(\frac{x''^2}{a^2}+\frac{y''^2}{b^2}+\frac{z''^2}{c^2}-1\right)$$
$$=\left(\frac{x'x''}{a^2}+\frac{y'y''}{b^2}+\frac{z'z''}{c^2}-1\right)^2.$$

Hence, if the point P (x', y', z') be fixed, the co-ordinates of any point Q, on any tangent line from P to the ellipsoid, must satisfy the equation

$$\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}+\frac{z'^2}{c^2}-1\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)$$
$$-\left(\frac{xx'}{a^2}+\frac{yy'}{b^2}+\frac{zz'}{c^2}-1\right)^2=0\,\ldots\ldots\ldots\text{(i)}.$$

Hence (i) is the required equation of the tangent cone from (x', y', z') to the ellipsoid.

112. If we suppose the point (x', y', z') to move to an infinite distance, the cone will become a cylinder whose generating lines are parallel to the line from the centre of the ellipsoid to the point (x', y', z').

Hence, if in the equation of the enveloping cone we put
$$x' = lr,\ y' = mr,\ z' = nr,$$
and then make r infinitely great, we shall obtain the equation of the enveloping cylinder whose generating lines are parallel to
$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}.$$

Substituting lr, mr, nr for x', y', z' respectively in the equation of the enveloping cone we have
$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2} - \frac{1}{r^2}\right) - \left(\frac{xl}{a^2} + \frac{ym}{b^2} + \frac{zn}{c^2} - \frac{1}{r}\right)^2 = 0.$$

Hence, when r is infinite,
$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right) - \left(\frac{xl}{a^2} + \frac{ym}{b^2} + \frac{zn}{c^2}\right)^2 = 0.$$

113. The equation of the enveloping cylinder can be found, independently of the enveloping cone, in the following manner.

The equations of the straight line which is drawn through any point (x', y', z') parallel to
$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n},$$
are
$$\frac{x - x'}{l} = \frac{y - y'}{m} = \frac{z - z'}{n} = r.$$

The straight line will meet the ellipsoid in two points whose distances from (x', y', z') are given by the equation
$$\left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{z'^2}{c^2} - 1\right) + 2r\left(\frac{lx'}{a^2} + \frac{my'}{b^2} + \frac{nz'}{c^2}\right) + r^2\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right) = 0.$$

The straight line will therefore touch the surface, if
$$\left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{z'^2}{c^2} - 1\right)\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right) = \left(\frac{lx'}{a^2} + \frac{my'}{b^2} + \frac{nz'}{c^2}\right)^2.$$

Hence the co-ordinates of any point, which is on a tangent line parallel to

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n},$$

satisfy the equation

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)-\left(\frac{lx}{a^2}+\frac{my}{b^2}+\frac{nz}{c^2}\right)^2=0,$$

which is the required equation of the enveloping cylinder.

Ex. (i). To find the condition that the enveloping cone may have three perpendicular generators.

The equation of the enveloping cone whose vertex is (x', y', z') is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}+\frac{z'^2}{c^2}-1\right)-\left(\frac{xx'}{a^2}+\frac{yy'}{b^2}+\frac{zz'}{c^2}-1\right)^2=0.$$

If this have three perpendicular generators the sum of the coefficients of x^2, y^2, and z^2 must be equal to zero [Art. 109]. Hence (x', y', z'), the vertex of the cone, is on the surface

$$\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)=\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}.$$

Ex. (ii). Shew that any two enveloping cones of an ellipsoid intersect in plane curves.

The equations of the cones whose vertices are (x', y', z') and (x'', y'', z'') are

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}+\frac{z'^2}{c^2}-1\right)=\left(\frac{xx'}{a^2}+\frac{yy'}{b^2}+\frac{zz'}{c^2}-1\right)^2,$$

and $$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)\left(\frac{x''^2}{a^2}+\frac{y''^2}{b^2}+\frac{z''^2}{c^2}-1\right)=\left(\frac{xx''}{a^2}+\frac{yy''}{b^2}+\frac{zz''}{c^2}-1\right)^2$$

respectively.

The surface whose equation is

$$\left(\frac{xx'}{a^2}+\frac{yy'}{b^2}+\frac{zz'}{c^2}-1\right)^2\left(\frac{x''^2}{a^2}+\frac{y''^2}{b^2}+\frac{z''^2}{c^2}-1\right)$$
$$=\left(\frac{xx''}{a^2}+\frac{yy''}{b^2}+\frac{zz''}{c^2}-1\right)^2\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}+\frac{z'^2}{c^2}-1\right)$$

passes through their common points, and clearly is two planes.

Ex. (iii). Find the equation of the enveloping cone of the paraboloid $ax^2+by^2+2z=0$.

Ans. $(ax^2+by^2+2z)(ax'^2+by'^2+2z')=(axx'+byy'+z+z')^2$.

Ex. (iv). Find the locus of a point from which three perpendicular tangent lines can be drawn to the paraboloid $ax^2+by^2+2z=0$.

Ans. $ab(x^2+y^2)+2(a+b)z=1$.

Examples on Chapter IV.

1. Find the equation of a sphere which cuts four given spheres orthogonally.

2. Shew that a sphere which cuts the two spheres $S=0$ and $S'=0$ at right angles, will cut $lS+mS'=0$ at right angles.

3. OP, OQ, OR are three perpendicular lines which meet in a fixed point O, and cut a given sphere in the points P, Q, R; shew that the locus of the foot of the perpendicular from O on the plane PQR is a sphere.

4. Through a point O two straight lines are drawn perpendicular to one another and intersecting two given straight lines at right angles; shew that the locus of O is a conicoid whose centre is the middle point of the shortest distance between the given lines.

5. Shew that the cone $Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0$ will have three of its generators coincident with conjugate diameters of $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, if $Aa^2+Bb^2+Cc^2=0$.

6. A plane moves so that the sum of the squares of its distances from n given points is constant; shew that it always touches an ellipsoid.

7. The normals to a surface of the second degree, at all points of a plane section parallel to a principal plane, meet two fixed straight lines, one in each of the other principal planes.

8. Shew that the plane joining the extremities of three conjugate diameters of an ellipsoid, touches another ellipsoid.

9. Having given any two systems of conjugate semi-diameters of an ellipsoid, the parallelopiped which has any three for conterminous edges is equal to that which has the other three for conterminous edges.

10. If lines be drawn through the centre of an ellipsoid parallel to the generating lines of an enveloping cone, the cone so formed will intersect the ellipsoid in two planes parallel to the plane of contact.

11. The enveloping cone from a point P to an ellipsoid has three generating lines parallel to conjugate diameters of the ellipsoid; find the locus of P.

12. The plane through the three points in which any three conjugate diameters of a conicoid meet the director-sphere touches the conicoid.

13. Shew that any two sets of three conjugate diameters of a conicoid are generators of a cone of the second degree.

14. Shew that any two sets of three conjugate diametral planes of a conicoid touch a cone of the second degree.

15. Shew that any one of three equal conjugates of an ellipsoid is on the cone whose equation is

$$(a^2+b^2+c^2)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)=3(x^2+y^2+z^2).$$

16. D, E, F and P, Q, R are the extremities of two sets of conjugate diameters of an ellipsoid. If p, p_1, p_2, p_3 are the perpendiculars from the centre and P, Q, R respectively on the plane DEF, prove that

$$p_1^2+p_2^2+p_3^2=2p\,(p_1+p_2+p_3).$$

17. The sum of the products of the perpendiculars from the two extremities of each of three conjugate diameters on any tangent plane to an ellipsoid is equal to twice the square on the perpendicular from the centre on that tangent plane.

18. The distance r is measured inwards along the normal to an ellipsoid at any point P, so that $pr=m^2$, where p is the perpendicular from the centre on the tangent plane at P; shew that the locus of the point so obtained is

$$\frac{a^2x^2}{(a^2-m^2)^2}+\frac{b^2y^2}{(b^2-m^2)^2}+\frac{c^2z^2}{(c^2-m^2)^2}=1.$$

19. Through any point P on an ellipsoid chords PQ, PR, PS are drawn parallel to the axes; find the equation of the plane QRS, and shew that the locus of K, the point of intersection of the plane QRS and the normal at P, is another ellipsoid. Shew also that if the normal at P meet the principal planes in G_1, G_2, G_3 then will

$$\frac{2}{PK}=\frac{1}{PG_1}+\frac{1}{PG_2}+\frac{1}{PG_3}.$$

20. PK is the perpendicular from any point on its polar plane with respect to a conicoid and this perpendicular meets a principal plane in G; shew that, if $PK.PG$ is constant, the locus of P is a conicoid.

21. Shew that the cone whose base is the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $z=0$, and whose vertex is any point of the hyperbola $\frac{x^2}{a^2-b^2}-\frac{z^2}{b^2}=1$, $y=0$, is a right circular cone.

22. A cone, whose equation referred to its principal axes, is

$$\alpha^2\xi^2+\beta^2\eta^2=(\alpha^2+\beta^2)\zeta^2,$$

is thrust into an elliptic hole whose equation is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; shew that when the cone fits the hole its vertex must lie on the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+z^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)=1.$$

23. In a cone any system of three conjugate diameters meets any plane section in the angular points of a triangle self polar with respect to that section.

24. The enveloping cones which have as vertices two points on the same diameter of a conicoid intersect in two parallel planes between whose distances from the centre that of the tangent plane at the end of the diameter is a mean proportional. What is the corresponding proposition for a paraboloid?

25. Shew that any two enveloping cones intersect in plane curves; and that when the planes are at right angles to one another, the product of the perpendiculars on one of the planes of contact from the centre of the ellipsoid and the vertex of the corresponding cone, is equal to the product of such perpendiculars on the other plane of contact.

26. If a line through a fixed point O be such that its conjugate line with respect to a conicoid is perpendicular to it, shew that the line is a generating line of a quadric cone.

27. The locus of the feet of the perpendiculars let fall from points on a given diameter of a conicoid on the polar planes of those points is a rectangular hyperbola.

28. Prove that the surfaces

$$\frac{x^2}{a_1^2}+\frac{y^2}{b_1^2}=\frac{2z}{c_1},\quad \frac{x^2}{a_2^2}+\frac{y^2}{b_2^2}=\frac{2z}{c_2},\quad \frac{x^2}{a_3^2}+\frac{y^2}{b_3^2}=\frac{2z}{c_3},$$

will have a common tangent plane if

$$\begin{vmatrix} a_1^2, & a_2^2, & a_3^2 \\ b_1^2, & b_2^2, & b_3^2 \\ c_1, & c_2, & c_3 \end{vmatrix}=0.$$

29. Prove that an ellipsoid of semi-axes a, b, c and a concentric sphere of radius $\dfrac{abc}{\sqrt{b^2c^2+c^2a^2+a^2b^2}}$, are so related that an indefinite number of octahedrons can be inscribed in the ellipsoid, and at the same time circumscribed to the sphere, the diagonals of the octahedrons intersecting at right angles in the centre.

30. Find the locus of the centre of sections of $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ which touch $\dfrac{x^2}{a'^2}+\dfrac{y^2}{b'^2}+\dfrac{z^2}{c'^2}=1$.

31. Planes are drawn through a given line so as to cut an ellipsoid; shew that the centres of the sections so formed all lie on a conic.

32. Find the locus of the centres of sections of an ellipsoid by planes which are at a constant distance from the centre.

33. Shew that the plane sections of an ellipsoid which have their centres on a fixed straight line are parallel to another straight line, and touch a parabolic cylinder.

34. The locus of the line of intersection of two perpendicular tangent planes to $ax^2+by^2+cz^2=0$ is

$$a(b+c)x^2+b(c+a)y^2+c(a+b)z^2=0.$$

35. The points on a conicoid the normals at which intersect the normal at a fixed point all lie on a cone of the second degree whose vertex is the fixed point.

36. Normals are drawn to a conicoid at points where it is met by a cone which has the axes of the conicoid for three of its generating lines; shew that all the normals intersect a fixed diameter of the conicoid.

37. Shew that the six normals which can be drawn from any point to an ellipsoid lie on a cone of the second degree, three of whose generating lines are parallel to the axes of the ellipsoid.

38. Find the equations of the right circular cylinders which circumscribe an ellipsoid.

39. If a right circular cone has three generating lines mutually at right angles, the semi-vertical angle is $\tan^{-1}\sqrt{2}$.

40. If one of the principal axes of a cone which stands on a given base be always parallel to a given right line, the locus of the vertex is an equilateral hyperbola or a right line according as the base is a central conic or a parabola.

41. The axis of the right circular cone, vertex at the origin, which passes through the three lines, whose direction-cosines are (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3) is normal to the plane

$$\begin{vmatrix} 0, & 1, & 1, & 1 \\ x, & l_1, & l_2, & l_3 \\ y, & m_1, & m_2, & m_3 \\ z, & n_1, & n_2, & n_3 \end{vmatrix} = 0.$$

42. The equations of the axes of the four cones of revolution which can be described touching the co-ordinate planes are

$$\frac{x^2}{\sin^2\alpha} = \frac{y^2}{\sin^2\beta} = \frac{z^2}{\sin^2\gamma}:$$

α, β, γ being the angles YOZ, ZOX, and XOY respectively.

43. Prove that four right cones may be described, passing through three given straight lines intersecting in the same point, and that if 2α, 2β, 2γ be the mutual inclinations of the straight lines, the equations of the cones referred to the straight lines as co-ordinate axes will be

$$\frac{\sin^2\alpha}{x} + \frac{\sin^2\beta}{y} + \frac{\sin^2\gamma}{z} = 0, \quad \frac{\sin^2\alpha}{x} + \frac{\cos^2\beta}{y} + \frac{\cos^2\gamma}{z} = 0,$$

$$-\frac{\cos^2\alpha}{x} + \frac{\sin^2\beta}{y} + \frac{\cos^2\gamma}{z} = 0, \quad -\frac{\cos^2\alpha}{x} - \frac{\cos^2\beta}{y} + \frac{\sin^2\gamma}{z} = 0.$$

44. Shew that, if P, Q, R be extremities of three conjugate diameters of a conicoid, the conic in which the plane PQR cuts the surface contains an infinite number of sets of three conjugate extremities, which are at the angular points of maximum triangles inscribed in the conic PQR.

45. Shew that, if the feet of three of the six normals drawn from any point to an ellipsoid lie on the plane $lx + my + nz + p = 0$, the feet of the other three will be on the plane

$$\frac{ax}{l} + \frac{by}{m} + \frac{cz}{n} - \frac{1}{p} = 0,$$

the equation of the ellipsoid being $ax^2 + by^2 + cz^2 = 1$.

46. Prove that the locus of a point with which as a centre of conical projection, a given conic on a given plane may be projected into a circle on another given plane, is a plane conic.

47. If C be the centre of a conicoid, and $P(Q)$ denote the perpendicular from P on the polar plane of Q; then will

$$\frac{P(Q)}{Q(P)} = \frac{C(Q)}{C(P)}.$$

48. The locus of a point such that the sum of the squares of its normal distances from a given ellipsoid is constant, is a co-axial ellipsoid.

49. If a line cut two similar and co-axial ellipsoids in P, P'; Q, Q'; prove that the tangent plane to the former at P, P', meet those to the latter at Q or Q' in pairs of parallel lines equidistant respectively from Q or Q'.

50. A chord of a quadric is intersected by the normal at a given point of the surface, the product of the tangents of the angles subtended at the point by the two segments of the chord being invariable. Prove that, O being the given point and P, P' the intersections of the normal with two such chords in perpendicular normal planes, the sum of the reciprocals of OP, OP', is invariable.

CHAPTER V.

Plane Sections of Conicoids.

114. We have seen [Art. 51] that all plane sections of a conicoid are conics, and also [Art. 61] that all parallel sections are similar conics. Since ellipses, parabolas, and hyperbolas are orthogonally projected into ellipses, parabolas, and hyperbolas respectively, we can find whether the curve of intersection of a conicoid and a plane is an ellipse, parabola, or hyperbola, by finding the equation of the projection of the section on one of the co-ordinate planes.

For example, to find the nature of plane sections of a paraboloid.

The plane $lx + my + nz + p = 0$ cuts the paraboloid $ax^2 + by^2 + 2z = 0$, in a curve through which the cylinder

$$a(my + nz + p)^2 + bl^2y^2 + 2l^2z = 0$$

passes. The plane $x = 0$, which is perpendicular to the generating lines of the cylinder, cuts it in the conic whose equations are $x = 0$, $a(my + nz + p)^2 + bl^2y^2 + 2l^2z = 0$; and this conic is the projection of the section on the plane $x = 0$. If $n = 0$, the projection will be a parabola; but, if n be not zero, the projection will be an ellipse or hyperbola according as $an^2(am^2 + bl^2) - a^2m^2n^2$ is positive or negative, or abl^2n^2 positive or negative, that is, according as the surface is an elliptic or hyperbolic paraboloid.

Hence all sections of a paraboloid which are parallel to the axis of the surface are parabolas; all other sections of an elliptic paraboloid are ellipses, and of a hyperbolic paraboloid are hyperbolas.

Ex. 1. Find the condition that the section of $ax^2+by^2+cz^2=1$ by the plane $lx+my+nz+p=0$ may be a parabola.

Ans. $\dfrac{l^2}{a}+\dfrac{m^2}{b}+\dfrac{n^2}{c}=0.$

Ex. 2. Shew that any tangent plane to the asymptotic cone of a conicoid meets the conicoid in two parallel straight lines.

115. *To find the axes and area of any central plane section of an ellipsoid.*

Let the equation of the ellipsoid be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

and let the equation of the plane be

$$lx+my+nz=0 \text{..................(i).}$$

Every semi-diameter of the surface whose length is r is a generating line of the cone whose equation is [p. 55, Ex. 5]

$$x^2\left(\frac{1}{a^2}-\frac{1}{r^2}\right)+y^2\left(\frac{1}{b^2}-\frac{1}{r^2}\right)+z^2\left(\frac{1}{c^2}-\frac{1}{r^2}\right)=0 \text{......(ii).}$$

This cone will, for all values of r, be cut by the plane in two straight lines which lie along equal diameters of the section; and, when r is equal to either semi-axis of the section, these equal diameters will coincide. That is, the plane (i) will touch the cone (ii) when r is equal to either semi-axis of the section of the ellipsoid by the plane. The condition of tangency gives

$$\frac{l^2}{\frac{1}{a^2}-\frac{1}{r^2}}+\frac{m^2}{\frac{1}{b^2}-\frac{1}{r^2}}+\frac{n^2}{\frac{1}{c^2}-\frac{1}{r^2}}=0 \text{.........(iii).}$$

From (iii) we see that

$$r_1r_2=\frac{abc}{\sqrt{(a^2l^2+b^2m^2+c^2n^2)}}=\frac{abc}{p} \text{............(iv),}$$

where r_1, r_2 are the semi-axes of the section, and p is the perpendicular on the parallel tangent plane.

From (iv) we see that the area of the section is equal to

$$\frac{\pi abc}{\sqrt{(a^2l^2+b^2m^2+c^2n^2)}}.$$

116. *To find the area of any plane section of an ellipsoid.*

Take for co-ordinate planes three conjugate planes of which $z=0$ is parallel to the given plane; then the equations of the surface and of the given plane will be respectively of the forms

$$\frac{x^2}{a'^2}+\frac{y^2}{b'^2}+\frac{z^2}{c'^2}=1, \text{ and } z=k.$$

The cylinder whose equation is

$$\frac{x^2}{a'^2}+\frac{y^2}{b'^2}+\frac{k^2}{c'^2}=1,$$

passes through the curve of intersection of the surface and the plane; and the area of the section of this cylinder by $z=k$ is

$$\pi a'b' \sin\nu\left(1-\frac{k^2}{c'^2}\right),$$

ν being the angle XOY. The area of the section of the ellipsoid by $z=0$ is $\pi a'b' \sin\nu$.

Hence, if A be the required area, and A_0 be the area of the parallel central section, we have

$$A=A_0\left(1-\frac{k^2}{c'^2}\right).$$

Now the tangent plane at $(0, 0, c')$ is $z=c'$; therefore the perpendicular distances of the given plane and of the parallel tangent plane from the centre are in the ratio of $k : c'$.

Hence $$A=A_0\left(1-\frac{p^2}{p_0^2}\right) \text{..................(i)},$$

where p and p_0 are the perpendicular distances of the given plane and of the parallel tangent plane from the centre.

This gives the relation between the area of any section and of the parallel central section; and we have found in Art. 115, the area of any central section.

Hence the area of the section of the ellipsoid whose equation, referred to its principal axes, is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

made by the plane whose equation is

$$lx+my+nz=p,$$

is
$$\frac{\pi abc}{\sqrt{(a^2l^2+b^2m^2+c^2n^2)}}\left(1-\frac{p^2}{a^2l^2+b^2m^2+c^2n^2}\right).$$

For
$$A_0=\frac{\pi abc}{\sqrt{(a^2l^2+b^2m^2+c^2n^2)}} \qquad \text{[Art. 115]},$$

and
$$p_0^2=a^2l^2+b^2m^2+c^2n^2 \qquad \text{[Art. 91]}.$$

Ex. 1. To find the area of the section of a paraboloid by any plane.

Let the equation of the paraboloid be $ax^2+by^2+2z=0$, and let the equation of the section be $lx+my+nz+p=0$. The projection of the section on the plane $z=0$ is the conic

$$ax^2+by^2-\frac{2}{n}(lx+my+p)=0,$$

or
$$a\left(x-\frac{l}{na}\right)^2+b\left(y-\frac{m}{nb}\right)^2=\frac{1}{n^2}\left(\frac{l^2}{a}+\frac{m^2}{b}+2pn\right).$$

The area of the projection is

$$\frac{\pi}{n^2\sqrt{ab}}\left(\frac{l^2}{a}+\frac{m^2}{b}+2pn\right);$$

and therefore [Art. 31] the area of the section is

$$\frac{\pi}{n^3\sqrt{ab}}\left\{\frac{l^2}{a}+\frac{m^2}{b}+2pn\right\}.$$

Ex. 2. To find the area of the section of the cone $\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=0$ by the plane $lx+my+nz=p$.

The area of the section of $\frac{x^2}{ak}+\frac{y^2}{bk}+\frac{z^2}{ck}=1$ by the given plane is

$$\frac{\pi\sqrt{abck^3}}{\sqrt{(kal^2+kbm^2+kcn^2)}}\left\{1-\frac{p^2}{kal^2+kbm^2+kcn^2}\right\}.$$

If we put $k=0$ the surface becomes the cone. The required area is therefore

$$\frac{\pi p^2\sqrt{abc}}{(al^2+bm^2+cn^2)^{\frac{3}{2}}}.$$

Ex. 3. If central plane sections of an ellipsoid be of constant area, their planes touch a cone of the second degree.

Let the area be $\frac{\pi abc}{d}$, and let the equation of one of the planes be

$$lx+my+nz=0.$$

Then we have

$$\frac{\pi abc}{\sqrt{(a^2l^2+b^2m^2+c^2n^2)}}=\frac{\pi abc}{d},$$

or
$$a^2l^2+b^2m^2+c^2n^2=d^2;$$

$\therefore$
$$(a^2-d^2)\,l^2+(b^2-d^2)\,m^2+(c^2-d^2)\,n^2=0.$$

This shews that the plane $lx+my+nz=0$ always touches the cone

$$\frac{x^2}{a^2-d^2}+\frac{y^2}{b^2-d^2}+\frac{z^2}{c^2-d^2}=0.$$

117. We can find, by the method of Art. 115, the area of a central plane section of the surface whose equation is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=1.$$

For the semi-diameters of length r are generating lines of the cone whose equation is

$$\left(a-\frac{1}{r^2}\right)x^2+\left(b-\frac{1}{r^2}\right)y^2+\left(c-\frac{1}{r^2}\right)z^2+2fyz+2gzx+2hxy=0.$$

When r is equal to either semi-axis of the section of the surface by the plane

$$lx+my+nz=0,$$

the plane will be a tangent plane of the cone. The condition of tangency gives, for the determination of the semi-axes, the equation

$$\begin{vmatrix} a-\frac{1}{r^2}, & h, & g, & l \\ h, & b-\frac{1}{r^2}, & f, & m \\ g, & f, & c-\frac{1}{r^2}, & n \\ l, & m, & n, & 0 \end{vmatrix}=0.$$

This result may also be obtained by finding the maximum value of $x^2+y^2+z^2\equiv r^2$, subject to the conditions $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=1$, and $lx+my+nz=0$.

118. *To find the directions of the axes of any central section of a conicoid.*

Let the equation of the surface be

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 1,$$

and let the equation of the plane be

$$lx + my + nz = 0.$$

Then, if P be any point on an axis of the section, the line joining P to the centre of the section will be perpendicular to the polar line of P in the plane of the section.

Hence, if P be (ξ, η, ζ), and if the direction-cosines of the polar line be λ, μ, ν, we have

$$\lambda\xi + \mu\eta + \nu\zeta = 0 \text{..................(i).}$$

Also, since the polar line is on both the planes

$$x(a\xi + h\eta + g\zeta) + y(h\xi + b\eta + f\zeta) + z(g\xi + f\eta + c\zeta) = 1,$$

and $$lx + my + nz = 0,$$

it is perpendicular to the normals to those planes; hence

$$\lambda(a\xi + h\eta + g\zeta) + \mu(h\xi + b\eta + f\zeta) + \nu(g\xi + f\eta + c\zeta) = 0 \text{...(ii),}$$

and $$\lambda l + \mu m + \nu n = 0 \text{..................(iii).}$$

Eliminating λ, μ, ν from the equations (i), (ii), (iii), we have

$$\begin{vmatrix} \xi, & \eta, & \zeta \\ a\xi + h\eta + g\zeta, & h\xi + b\eta + f\zeta, & g\xi + f\eta + c\zeta \\ l, & m, & n \end{vmatrix} = 0.$$

Hence the required axes are the lines in which the given plane cuts the cone whose equation is

$$\begin{vmatrix} x, & y, & z \\ ax + hy + gz, & hx + by + fz, & gx + fy + cz \\ l, & m, & n \end{vmatrix} = 0.$$

119. *To find the angle between the asymptotes of any plane section of a conicoid.*

Let θ be the angle between the asymptotes of the plane section, and let the semi-axes of the section be α, β.

Then $$\tan\frac{\theta}{2} = \sqrt{-1}\,\frac{\beta}{\alpha};$$

$$\therefore \qquad \tan^2\theta = \frac{-4\alpha^2\beta^2}{(\alpha^2+\beta^2)^2}.$$

This gives the required angle, since we have found, in the preceding articles, the axes of any plane section.

Ex. 1. Find the angle between the asymptotes of the section of $ax^2+by^2+cz^2=1$ by the plane $lx+my+nz=0$.

The semi-axes are the roots of the equation

$$\frac{l^2}{a-\frac{1}{r^2}} + \frac{m^2}{b-\frac{1}{r^2}} + \frac{n^2}{c-\frac{1}{r^2}} = 0;$$

therefore $$\tan^2\theta = -\frac{4r_1^2r_2^2}{(r_1^2+r_2^2)^2} = \frac{-4abc\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right)}{\{l^2(b+c)+m^2(c+a)+n^2(a+b)\}^2}.$$

Ex. 2. To find the condition that the section of the conicoid

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=1$$

by the plane $lx+my+nz=0$ may be a rectangular hyperbola.

The square of the reciprocal of the semi-diameter whose direction-cosines are λ, μ, ν is given by

$$\frac{1}{r^2} = a\lambda^2+b\mu^2+c\nu^2+2f\mu\nu+2g\nu\lambda+2h\lambda\mu.$$

Take any three perpendicular diameters; then we have by addition

$$\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2} = a+b+c.$$

Now, if r_1, r_2 be the lengths of any two perpendicular semi-diameters of a rectangular hyperbola, $r_1^2+r_2^2=0$.

Hence for any semi-diameter of the conicoid which is perpendicular to the plane of a section which is a rectangular hyperbola, we have

$$\frac{1}{r^2} = a+b+c.$$

The required condition is therefore

$$al^2+bm^2+cn^2+2fmn+2gnl+2hlm = a+b+c = (a+b+c)(l^2+m^2+n^2).$$

Ex. 3. Shew that the two lines given by the equations $ax^2+by^2+cz^2=0$, $lx+my+nz=0$ will be at right angles, if

$$l^2(b+c)+m^2(c+a)+n^2(a+b)=0.$$

The lines are the asymptotes of the section of the conicoid $ax^2+by^2+cz^2=1$ by the plane $lx+my+nz=0$.

120. *If two conicoids have one plane section in common all their other points of intersection lie on another plane.*

Let the equations of the common plane section be

$$ax^2 + by^2 + 2hxy + 2ux + 2vy + c = 0, \ z = 0.$$

The most general equations of two conicoids which pass through this conic are

$$ax^2 + by^2 + 2hxy + 2ux + 2vy + c + z(lx + my + nz + p) = 0,$$

and

$$ax^2 + by^2 + 2hxy + 2ux + 2vy + c + z(l'x + m'y + n'z + p') = 0.$$

It is clear that all points which are on both surfaces, and for which z is not zero, are on the plane given by the equation

$$lx + my + nz + p = l'x + m'y + n'z + p';$$

this proves the proposition.

Circular Sections.

121. *To find the circular sections of an ellipsoid.*

Since parallel sections are similar, we need only consider the sections through the centre.

Now all the semi-diameters of the ellipsoid which are of length r are generating lines of the cone whose equation is

$$x^2\left(\frac{1}{a^2} - \frac{1}{r^2}\right) + y^2\left(\frac{1}{b^2} - \frac{1}{r^2}\right) + z^2\left(\frac{1}{c^2} - \frac{1}{r^2}\right) = 0.$$

If there be a circular section of radius r, an infinite number of generating lines of the cone will lie on the plane of the section; hence the cone must be two planes. This will only be the case when r is equal to a, or b, or c.

If $r = a$, the two planes pass through the axis of x, their equation being

$$y^2\left(\frac{1}{b^2} - \frac{1}{a^2}\right) + z^2\left(\frac{1}{c^2} - \frac{1}{a^2}\right) = 0 \text{............(i)}.$$

The equations of the other pairs of planes are respectively

$$z^2\left(\frac{1}{c^2}-\frac{1}{b^2}\right)+x^2\left(\frac{1}{a^2}-\frac{1}{b^2}\right)=0\ldots\ldots\ldots\ldots(\text{ii}),$$

and $$x^2\left(\frac{1}{a^2}-\frac{1}{c^2}\right)+y^2\left(\frac{1}{b^2}-\frac{1}{c^2}\right)=0\ldots\ldots\ldots\ldots(\text{iii}).$$

Of these three pairs of planes, two are imaginary. For, if a, b, c be in order of magnitude, $\frac{1}{b^2}-\frac{1}{a^2}$ and $\frac{1}{c^2}-\frac{1}{a^2}$ have the same sign, and therefore the planes (i) are imaginary; for a similar reason the planes (iii) are imaginary. Hence, the only real central circular sections of an ellipsoid pass through the mean axis, and their equations are

$$x\sqrt{\left(\frac{1}{a^2}-\frac{1}{b^2}\right)}=\pm z\sqrt{\left(\frac{1}{b^2}-\frac{1}{c^2}\right)}\ldots\ldots(\text{iv}).$$

Since all parallel sections are similar, there are two systems of planes which cut the ellipsoid in circles, namely planes parallel to those given by the equation (iv).

If $b=c$ the two planes which give circular sections are coincident.

122. If the surface be an hyperboloid of one sheet, we must change the sign of c^2 in the equations of the last Article. In this case the planes which give the real circular sections are those given by equations (i), a being supposed to be greater than b.

If the surface be an hyperboloid of two sheets, we must change the signs of b^2 and c^2. In this case the planes which give the real circular sections are those given by equation (ii), b being supposed to be numerically greater than c.

123. If a series of planes be drawn parallel to either of the central circular sections of an ellipsoid, these planes will cut the surface in circles which become smaller and smaller as the planes are drawn farther and farther from the centre; and, when the plane is drawn so as to touch the ellipsoid, the circle will be indefinitely small.

DEF. The point of contact of a tangent plane which cuts a surface in a point-circle is called an *umbilic*.

124. *Any two circular sections of opposite systems are on a sphere.*

The circular sections of the ellipsoid are parallel to the planes whose equations are

$$x^2\left(\frac{1}{a^2}-\frac{1}{b^2}\right)+z^2\left(\frac{1}{c^2}-\frac{1}{b^2}\right)=0.$$

Hence $$x\sqrt{\left(\frac{1}{a^2}-\frac{1}{b^2}\right)}+z\sqrt{\left(\frac{1}{b^2}-\frac{1}{c^2}\right)}+p=0,$$

and $$x\sqrt{\left(\frac{1}{a^2}-\frac{1}{b^2}\right)}-z\sqrt{\left(\frac{1}{b^2}-\frac{1}{c^2}\right)}+q=0,$$

are the equations of the planes of any two circular sections of opposite systems.

The equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1-\lambda\left\{x\sqrt{\left(\frac{1}{a^2}-\frac{1}{b^2}\right)}+z\sqrt{\left(\frac{1}{b^2}-\frac{1}{c^2}\right)}+p\right\}$$
$$\left\{x\sqrt{\left(\frac{1}{a^2}-\frac{1}{b^2}\right)}-z\sqrt{\left(\frac{1}{b^2}-\frac{1}{c^2}\right)}+q\right\}=0,$$

is, for all values of λ, the equation of a conicoid which passes through the two circular sections; and, if $\lambda=1$, the equation represents a sphere; which proves the proposition.

125. We can find the circular sections of the paraboloid

$$\frac{x^2}{a}+\frac{y^2}{b}=2z,$$

by writing the equation in the form

$$\frac{1}{a}(x^2+y^2+z^2-2az)+y^2\left(\frac{1}{b}-\frac{1}{a}\right)-\frac{z^2}{a}=0.$$

It is clear that the two planes given by the equation

$$y^2\left(\frac{1}{b}-\frac{1}{a}\right)-\frac{z^2}{a}=0,$$

cut the paraboloid where they cut the sphere whose equation is $$x^2+y^2+z^2-2az=0;$$

and, since the planes must cut the sphere in circles, they will cut the paraboloid in circles.

We can shew in a similar manner that the planes given by the equation

$$x^2\left(\frac{1}{a}-\frac{1}{b}\right)-\frac{z^2}{b}=0,$$

will give circular sections of the paraboloid.

Of the two pairs of planes given by the equations

$$x^2\left(\frac{1}{a}-\frac{1}{b}\right)-\frac{z^2}{b}=0, \text{ and } y^2\left(\frac{1}{b}-\frac{1}{a}\right)-\frac{z^2}{a}=0,$$

one will be real, if a and b are of the same sign; but both pairs of planes will be imaginary if a and b are of different signs, so that there are no circular sections of a hyperbolic paraboloid.*

Ex. 1. Shew that the conicoid whose equation is

$$(A+\lambda)\,x^2+(B+\lambda)\,y^2+(C+\lambda)\,z^2=1,$$

has the same cyclic planes for all values of λ.

Ex. 2. Shew that no two parallel circular sections of a conicoid, which is not a surface of revolution, are on a sphere.

Ex. 3. Find the circular sections of the conicoid whose equation is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=1.$$

All semi-diameters which are of length r are generating lines of the cone whose equation is

$$\left(a-\frac{1}{r^2}\right)x^2+\left(b-\frac{1}{r^2}\right)y^2+\left(c-\frac{1}{r^2}\right)z^2+2fyz+2gzx+2hxy=0\ldots\text{(i)}.$$

If therefore r is the radius of a circular section, the cone must be two planes. The condition for this is

$$\begin{vmatrix} a-\frac{1}{r^2}, & h, & g \\ h, & b-\frac{1}{r^2}, & f \\ g, & f, & c-\frac{1}{r^2} \end{vmatrix}=0\ldots\ldots\ldots\ldots\ldots\text{(ii)}.$$

If we substitute in (i) any one of the roots of the equation (ii), we shall obtain the equation of the corresponding planes of circular section.

Ex. 4. Find the real circular sections of the following surfaces

(i) $4x^2+2y^2+z^2+3yz+zx=1$,

(ii) $2x^2+5y^2-3z^2+4xy=1$.

* This is not strictly true: a section through any generating line by a plane parallel to the axis of the surface is a circle of infinite radius.

Ans. (i) planes parallel to

$$(x+y-z)(x-y+2z)=0.$$

(ii) planes parallel to

$$(x+2y)^2-4z^2=0.$$

Ex. 5. Find the conditions that the plane

$$lx+my+nz=0,$$

may cut the conicoid

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=1$$

in a circle.

As in Ex. 3, the equation

$$\left(a-\frac{1}{\gamma^2}\right)x^2+\left(b-\frac{1}{\gamma^2}\right)y^2+\left(c-\frac{1}{\gamma^2}\right)z^2+2fyz+2gzx+2hxy=0$$

must, for some value of γ, be two planes of which the given plane is one. The equation must therefore be the same as

$$(lx+my+nz)\left\{\frac{x}{l}\left(a-\frac{1}{\gamma^2}\right)+\frac{y}{m}\left(b-\frac{1}{\gamma^2}\right)+\frac{z}{n}\left(c-\frac{1}{\gamma^2}\right)\right\}=0.$$

By comparing the coefficients of yz, zx, xy we have

$$\frac{m}{n}\left(c-\frac{1}{\gamma^2}\right)+\frac{n}{m}\left(b-\frac{1}{\gamma^2}\right)=2f,$$

and two similar equations.

Hence the required conditions are

$$\frac{bn^2+cm^2-2fmn}{m^2+n^2}=\frac{cl^2+an^2-2gnl}{n^2+l^2}=\frac{am^2+bl^2-2hlm}{l^2+m^2}.$$

126. We will conclude this chapter by the solution of two examples.

Ex. 1. *With a fixed point O on a conicoid as vertex, and plane sections of the conicoid for bases, cones are described; shew that the cones are cut by any plane parallel to the tangent plane at O in a system of similar conics.* (Chasles.)

The equation of a conicoid, referred to three conjugate diameters as axes, is of the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Hence the equation, referred to parallel axes through the extremity of one of the diameters, will be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}+\frac{2z}{c}=0.$$

This we will take for the equation of the surface, the common vertex of the cones being the origin. Let $lx+my+nz=1$ be the equation of any plane section; then the corresponding cone will be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}+\frac{2z}{c}(lx+my+nz)=0.$$

The section of this cone by the plane $z=k$ is clearly similar to the conic

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

which proves the proposition.

Ex. 2. *With a fixed point O on a conicoid for vertex, and a plane section of the conicoid for base, a cone is described; shew* (i) *that if the cone have three perpendicular generating lines, the plane base will meet the normal at* O *in a fixed point; and* (ii) *that if the normal at* O *be an axis of the cone, the plane base will meet the tangent plane at* O *in a fixed straight line.*

The most general equation of a conicoid, when the origin is on the surface and the plane $z=0$ is the tangent plane at the origin, is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2z=0.$$

The equation of the cone whose vertex is the origin, and which passes through the points of intersection of the conicoid and the plane

$$lx+my+nz=1$$

is $$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2z(lx+my+nz)=0.$$

Now the condition that the cone may have three perpendicular generating lines is

$$a+b+c+2n=0 \qquad \text{[Art. 109].}$$

This shews that the intercept on the axis of z is constant; which proves (i). The conditions that the axis of z may be an axis of the cone are [See Art. 60] $g+l=0$, and $f+m=0$. Hence the plane meets the axes of x and y in fixed points; which proves (ii).

Examples on Chapter V.

1. Shew that the area of the section of an ellipsoid, by a plane which passes through the extremities of three conjugate diameters, is in a constant ratio to the area of the parallel central section.

2. Given the sum of the squares of the axes of a plane central section of a conicoid, find the cone generated by a normal to its plane.

3. Shew that a plane which cuts off a constant volume from a cone envelopes a conicoid of which the cone is the asymptotic cone.

4. Shew that the axes of plane sections of the conicoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

which pass through the line

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$$

lie on the cone whose equation is

$$\frac{1}{x^2}\left(\frac{m}{y}-\frac{n}{z}\right)\left(\frac{1}{b^2}-\frac{1}{c^2}\right)+\frac{1}{y^2}\left(\frac{n}{z}-\frac{l}{x}\right)\left(\frac{1}{c^2}-\frac{1}{a^2}\right)+\frac{1}{z^2}\left(\frac{l}{x}-\frac{m}{y}\right)\left(\frac{1}{a^2}-\frac{1}{b^2}\right)=0.$$

5. If through a given point (x_0, y_0, z_0) lines be drawn each of which is an axis of some plane section of $ax^2+by^2+cz^2=1$, such lines describe the cone

$$a(b-c)\frac{x_0}{x-x_0}+b(c-a)\frac{y_0}{y-y_0}+c(a-b)\frac{z_0}{z-z_0}=0.$$

6. If the area of the section of

$$\frac{y^2}{b}+\frac{z^2}{c}=2x$$

be constant and equal to a^2, the locus of the centre is

$$a^4\left(1+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)^{-1}=\pi^2bc\left(2x-\frac{y^2}{b}-\frac{z^2}{c}\right)^2.$$

7. If a conic section, whose plane is perpendicular to a generator of a cone, be a circle; the corresponding projection of the reciprocal cone is a parabola.

8. Shew that the principal semi-axes of the normal section of the cylinder which envelopes $b^2c^2x^2+c^2a^2y^2+a^2b^2z^2=a^2b^2c^2$, and whose generating lines are parallel to

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n},$$

are the values of r given by

$$\frac{l^2}{a^2-r^2}+\frac{m^2}{b^2-r^2}+\frac{n^2}{c^2-r^2}=0.$$

9. Shew that the section of

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = \frac{2x}{a}$$

by the plane $lx + my + nz = 0$ is a rectangular hyperbola, if

$$(b^2 - c^2)\, l^2 + m^2b^2 - n^2c^2 = 0.$$

10. Shew that all plane sections of

$$\frac{x^2}{a} - \frac{y^2}{b} = z$$

which are rectangular hyperbolas, and which pass through the point (α, β, γ), touch the cone

$$\frac{(x-\alpha)^2}{a} - \frac{(y-\beta)^2}{b} + \frac{(z-\gamma)^2}{a-b} = 0.$$

11. Find the locus of the vertices of all parabolic sections of a paraboloid, whose planes are at the same distance from its axis.

12. Shew that, if the plane $lx + my + nz = p$ cut the surface $ax^2 + by^2 + cz^2 = 1$ in a parabola, the co-ordinates of the vertex of the parabola satisfy the equation

$$\frac{ax}{l}\left(\frac{1}{b} - \frac{1}{c}\right) + \frac{by}{m}\left(\frac{1}{c} - \frac{1}{a}\right) + \frac{cz}{n}\left(\frac{1}{a} - \frac{1}{b}\right) = 0.$$

13. The area of the section of $(abcfgh \between xyz)^2 = 1$ by the plane which passes through the extremities of its principal axes is

$$\frac{2\pi}{3\sqrt{3}} \sqrt{\left(\frac{a+b+c}{\Delta}\right)}.$$

14. A cone is described with vertex (f, g, h) and base the section of the surface $ax^2 + by^2 + cz^2 = 1$ made by the plane $x = 0$; shew that the equation of the plane in which this cone again meets the surface is

$$x\,(af^2 + bg^2 + ch^2 - 1) = 2f\,(afx + bgy + chz - 1).$$

15. Shew that the foci of all parabolic sections of

$$\frac{y^2}{a} + \frac{z^2}{b} = x,$$

lie on the surface

$$\left(x - \frac{y^2}{a} - \frac{z^2}{b}\right)\left(\frac{y^2}{a} + \frac{z^2}{b}\right) = \frac{ab}{4}\left(\frac{y^2}{a^2} + \frac{z^2}{b^2}\right).$$

16. Circles are described on a series of parallel chords of a fixed circle whose planes are inclined at a constant angle to the plane of the fixed circle.

Shew that they trace out an ellipsoid, the square on whose mean axis is an arithmetic mean between the squares on the other two axes.

17. Shew that if the squares of the axes of an ellipsoid are in arithmetical progression the umbilici lie on the central circular sections; if they are in harmonic progression the circular sections are at right angles; if they are in geometrical progression the tangent planes at the umbilici touch the sphere through the central circular sections.

18. Points on an ellipsoid such that the product of their distances from the two central circular sections is constant lie on the intersection of the ellipsoid with a sphere.

19. If the diameter of the sphere which passes through two circular sections of an ellipsoid be equal to its mean diameter, the distances of the planes from the centre are in a constant ratio.

20. A sphere of constant radius cuts an ellipsoid in plane curves; find the surface generated by their line of intersection.

21. The hyperboloid $x^2 + y^2 - z^2 \tan^2 \alpha = a^2$ is built up of thin circular discs of cardboard, strung by their centres on a straight wire. Prove that, if the wire be turned about the origin into the direction (l, m, n), the planes of the discs being kept parallel to their original direction, the equation of the surface will be

$$(nx - lz)^2 + (ny - mz)^2 = n^2 (z^2 \tan^2 \alpha + a^2).$$

22. If a series of parallel plane sections of an ellipsoid be taken, and on any sections as base a right cylinder be erected, shew that the other plane section, in which it meets the ellipsoid, will meet the plane of the base in a straight line whose locus will be a diametral plane of the ellipsoid.

23. Any number of similar and similarly situated conics, which are on a plane, are the stereographic projections of plane sections of some conicoid.

24. The tangent plane at an umbilicus meets any enveloping cone in a conic of which the umbilicus is a focus and the intersection of the plane of contact and the tangent plane a directrix.

25. The quadric $ax^2 + by^2 + cz^2 = 1$ is turned about its centre until it touches $a'x^2 + b'y^2 + c'z^2 = 1$ along a plane section. Find the equation to this plane section referred to the axes of either of the quadrics, and shew that its area is

$$\pi \sqrt{\frac{a + b + c - a' - b' - c'}{abc - a'b'c'}}.$$

CHAPTER VI.

GENERATING LINES OF CONICOIDS.

127. In cones and cylinders we have met with examples of curved surfaces on which straight lines can be drawn which will coincide with the surface throughout their entire length.

We shall in the present chapter shew that hyperboloids of one sheet, and hyperbolic paraboloids, can be generated by the motion of a straight line; and we shall investigate properties of those surfaces connected with the straight lines which lie upon them.

DEF. A surface through every point of which a straight line can be drawn so as to lie entirely on the surface, is called a *ruled surface;* and the straight lines which lie upon it are called *generating lines.*

A ruled surface on which consecutive generating lines intersect, is called a *developable surface.*

A ruled surface on which consecutive generating lines do not intersect, is called a *skew surface.*

128. To find where the straight line, whose equations are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r,$$

meets the surface whose equation is $F(x, y, z)=0$, we must substitute $\alpha+lr$, $\beta+mr$, and $\gamma+nr$ for x, y, z respectively, and we obtain the equation $F(\alpha+lr, \beta+mr, \gamma+nr)=0$.

If the surface is of the k^{th} degree, the equation for finding r is of the k^{th} degree; hence any straight line meets a surface of the k^{th} degree in k points.

If, however, for any particular straight line, *all* the coefficients in the equation for r are zero, that equation will be satisfied for all values of r; and therefore every point on that straight line will be on the surface. Since there are $k+1$ terms in the equation of the k^{th} degree, it follows that $k+1$ conditions must be satisfied in order that a straight line may lie entirely on a surface of the k^{th} degree.

Now the general equations of a straight line contain *four* independent constants, and therefore a straight line can be made to satisfy four conditions, and no more.

It follows therefore, that, if the degree of a surface be higher than the third, no straight line will, in general, lie altogether on the surface. For special forms of the equations of the fourth, or higher orders, we may however have generating lines; for example, the line whose equations are $y = mx$ and $z = m^3$ will, for all values of m, lie entirely on the surface whose equation is $zx^3 = y^3$.

If the equation of a surface be of the third degree, the number of conditions to be satisfied is equal to the number of constants in the general equations of a straight line. Hence the conditions can be satisfied, and there will be a *finite* number of solutions. The actual number of straight lines (real or imaginary) which lie on any cubic surface is 27. [See *Cambridge and Dublin Math. Journal*, Vol. IV.]

The number of conditions to be satisfied, in order that a straight line may lie entirely on a conicoid, is three. Since the number of conditions is less than the number of constants in the general equations of a straight line, the conditions can be satisfied in an infinite number of ways, so that there are an *infinite* number of generating lines on a conicoid; these generating lines may however all be imaginary, as is obviously the case when the surface is an ellipsoid.

129. A generating line on any surface touches the surface at any point O of its length, for it passes through a

point of the surface indefinitely near to O; hence the tangent plane to any surface at a point through which a generating line passes will contain that generating line.

130. The section of a conicoid by the tangent plane at any point through which a generating line passes, will be a conic of which the generator forms a part; the conic must therefore be two straight lines.

Hence, through any point on a generating line of a conicoid another generating line passes, and they are both in the tangent plane at the point.

The two generating lines in which the tangent plane to a conicoid intersects the surface are coincident when the conicoid is a cone or a cylinder.

131. Since any plane section of a conicoid is a conic, any plane which passes through a generating line of a conicoid will cut the surface in another generating line; and both generating lines are in the tangent plane at their point of intersection. Hence, *any plane through a generating line of a conicoid touches the surface,* its point of contact being the point of intersection of the two generating lines which lie upon it.

132. *To find which of the conicoids are ruled surfaces.*

If a conicoid have one generating line upon it, and we draw a plane through that generating line and any point P of the surface, this plane will cut the surface in another generating line, which must pass through P.

Hence, if there be a single generating line on a conicoid, there will be one, and therefore by Art. 130, two generating lines, through *every* point on the surface.

We can therefore at once determine whether a conicoid is or is not a ruled surface, by finding the nature of the intersection of the surface by the tangent plane at any particular point.

The equation of the tangent plane at the point $(a, 0, 0)$ of the conicoid $\frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 1$ is $x = a$; this meets the surface in straight lines whose projection on the plane $x=0$ are given by the equation $\pm\frac{y^2}{b^2}\pm\frac{z^2}{c^2}=0$. These lines are clearly real when the surface is an hyperboloid of one sheet, and imaginary when the surface is an ellipsoid, or an hyperboloid of two sheets.

Hence the hyperboloid of one sheet is a ruled surface.

The hyperbolic paraboloid is a particular case of the hyperboloid of one sheet; hence the hyperbolic paraboloid is also a ruled surface.

This can be proved at once from the equation of the paraboloid. For, the tangent plane at the origin is $z=0$, and this meets the paraboloid $ax^2+by^2+2z=0$ in the straight lines given by the equations $ax^2+by^2=0$, $z=0$; the lines are clearly real when a and b have different signs, and are imaginary when a and b have the same sign.

Hence an hyperboloid of one sheet (including an hyperbolic paraboloid as a particular case) is the only ruled conicoid in addition to a cone, a cylinder, and a pair of planes.

133. *To shew that there are two systems of generating lines on an hyperboloid of one sheet.*

Since any plane meets any straight line, the tangent plane at any point P on an hyperboloid of one sheet will meet all the generating lines of the surface, and the points of intersection will be on the surface. But the tangent plane cuts the surface in the two generating lines through P; hence every generating line of the hyperboloid must intersect one or other of the two generators PA, PB which pass through any point P on the surface.

Now no two of the generating lines which meet the same generator can themselves intersect, for otherwise there would be three generating lines in a plane, which is impossible, since every plane section is a conic.

Hence there are two systems of generating lines, which are such that all the members of one system intersect PB, but do not themselves intersect; and all the members of the other system intersect PA, but do not themselves intersect. Since the position of P is arbitrary it follows that every member of one of the two systems of generating lines meets every member of the other system.

134. If a straight line intersect a conicoid in *three* points, it will entirely coincide with the surface; and hence, to have a generating line of a conicoid given, is equivalent to having *three* points given.

To have three non-intersecting generating lines given is therefore equivalent to having *nine* points given, so that [Art. 50] three non-intersecting generators are sufficient to determine the conicoid on which they lie.

If a line meet three non-intersecting lines, it will meet the conicoid of which they are generators in three points, namely in the three points in which it intersects the three lines; and hence it must itself be a generator of the surface. Hence, the straight lines which intersect three fixed non-intersecting straight lines are generators of the same system of a conicoid, and the three fixed lines are generators of the opposite system of the same conicoid. [See Art. 49, Ex. 2]

135. Since any line which meets three non-intersecting straight lines is a generating line of the conicoid on which they lie, it follows that the only lines which meet the three lines and which also meet a fourth given straight line are the generators of the surface, of the system opposite to that defined by the given lines, which pass through the points where the conicoid is met by the fourth given straight line. But the fourth straight line will meet the conicoid in two points only, unless it be itself a generator of the surface.

Hence *two* straight lines, and two only, will, in general, meet each of four given non-intersecting straight lines; but if the four given straight lines are all generators of the same system of a conicoid, then an infinite number of straight lines will meet the four, which will all be generators of the opposite system of the same conicoid.

Ex. 1. Two planes are drawn, one through each of two intersecting generating lines of a conicoid; shew that the planes meet the surface in two other intersecting generating lines.

Ex. 2. Shew that the plane through the centre of a conicoid and any generating line, will cut the surface in a parallel generating line, and will touch the asymptotic cone.

Ex. 3. A conicoid is described to pass through two non-intersecting given lines and to touch a given plane. Shew that the locus of the point of contact is a straight line.

Let the given lines meet the given plane in the points A, B respectively. Then, the given plane will cut the surface in two generating lines, one of which will intersect both the given lines; hence, since the points of intersection must be A and B, the point of contact must be on the line AB.

Ex. 4. The lines through the angular points of a tetrahedron perpendicular to the opposite faces are generators of the same system of a conicoid.

Let AA', BB', CC', DD' be the four perpendiculars, and let α, β, γ, δ be the orthocentres of the faces opposite to A, B, C, D respectively. Then, it is easy to prove that the lines through α, β, γ, δ parallel respectively to AA', BB', CC', DD' will meet all the four perpendiculars. Since the four perpendiculars are met by more than two straight lines, they are generators of the same system of a conicoid; and the four parallel lines through α, β, γ, δ are generators of the opposite system of the same conicoid.

Ex. 5. If a rectilineal quadrilateral $ABCD$ be traced on a conicoid, the centre of the surface is on the straight line which passes through the middle points of the diagonals AC, BD.

The planes BAD, BCD are the tangent planes at A, C respectively, and BD is their line of intersection; hence the centre of the conicoid is on the plane through BD and the middle point of AC. Similarly the centre is on the plane through AC and the middle point of BD.

Ex. 6. If a rectilineal hexagon be traced on a conicoid, the three lines joining opposite vertices will meet in a point, and the three lines of intersection of the tangent planes at opposite vertices lie in a plane. [Dandelin.]

Let $ABCDEF$ be the hexagon. Intersecting generators of a conicoid are of different systems; therefore AB, CD, EF are of one system, and BC, DE, FA of the opposite system; so that opposite sides of the hexagon are of different systems, and therefore will intersect. Each of the diagonals AD, BE, CF is the line of intersection of two of the planes through pairs of opposite sides; therefore AD, BE, CF meet in a point, namely in the point of intersection of the three planes through pairs of opposite sides.

Let X be the point of intersection of AB and DE, Y the point of intersection of BC and EF, and Z of CD and FA. The tangent planes at A, D, namely the planes FAB, CDE, intersect in the line XZ; the tangent planes at B, E intersect in the line XY; and the tangent planes at C, F intersect in the line YZ. Hence the three lines of intersection of the tangent planes at opposite vertices lie in the plane XYZ.

Ex. 7. Four fixed generators of the same system cut all generators of the opposite system in a range of constant cross-ratio. [Chasles.]

Let any three generators of the opposite system cut the fixed generators in the points A, B, C, D; A', B', C', D' and A'', B'', C'', D'' respectively. Then, the four planes through $A''B''C''D''$ and the fixed generators cut all other straight lines in a range of constant cross-ratio [Art. 36]; we therefore have

$$\{A'B'C'D'\} = \{ABCD\}.$$

Ex. 8. The lines joining corresponding points of two homographic systems, on two given straight lines, are generating lines of a conicoid.

136. *To find the angle between the two generating lines through any point of an hyperboloid.*

The section of an hyperboloid of one sheet by the tangent plane at any point is similar and similarly situated to the parallel central section. Hence the generating lines through any point are parallel to the asymptotes of the parallel central section. Let the equation of the surface be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

and let f, g, h be the co-ordinates of the point P through which the generating lines pass.

Let α^2, β^2 be the squares of the axes of the central section which is parallel to the tangent plane at P, and let θ be the angle between the generating lines through P.

Then $$\tan\frac{\theta}{2} = \sqrt{-1}\,\frac{\beta}{\alpha},$$

and therefore

$$\tan\theta = 2\sqrt{-1}\,\frac{\alpha\beta}{\alpha^2 + \beta^2}.$$

Now the sum of the squares of three conjugate semi-diameters is constant, and also the parallelopiped of which they are conterminous edges. Hence

$$\alpha^2 + \beta^2 + OP^2 = a^2 + b^2 - c^2,$$

and $$\alpha\beta p = \sqrt{-1}\,.\,abc.$$

Hence we have

$$\tan\theta = 2\frac{abc}{p\,(a^2 + b^2 - c^2 - OP^2)}.$$

137. We can write the equation of an hyperboloid of one

sheet in such a way as to shew at once the existence of generating lines. For, the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,$$

is equivalent to

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=1-\frac{y^2}{b^2};$$

and it is evident that all points on the line of intersection of the planes whose equations are

$$\frac{x}{a}-\frac{z}{c}=\lambda\left(1-\frac{y}{b}\right),\ \frac{x}{a}+\frac{z}{c}=\frac{1}{\lambda}\left(1+\frac{y}{b}\right)$$

are on the surface; and by giving different values to λ we obtain a system of straight lines which lie altogether on the surface. The generating lines of the other system are similarly given by the equations

$$\frac{x}{a}-\frac{z}{c}=\lambda\left(1+\frac{y}{b}\right),\ \frac{x}{a}+\frac{z}{c}=\frac{1}{\lambda}\left(1-\frac{y}{b}\right).$$

We can find in a similar manner the equations of the generating lines of the paraboloid

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z.$$

The equations of the generators of one system are

$$\frac{x}{a}-\frac{y}{b}=2\lambda z,\ \ \frac{x}{a}+\frac{y}{b}=\frac{1}{\lambda};$$

and of the other system

$$\frac{x}{a}+\frac{y}{b}=2\lambda z,\ \frac{x}{a}-\frac{y}{b}=\frac{1}{\lambda}.$$

138. The equations of the generating lines which pass through any point on an hyperboloid of one sheet can be obtained in the following manner.

The co-ordinates of any point on the surface can be expressed in terms of two variables θ and ϕ, where

$$x=a\cos\theta\sec\phi,\ y=b\sin\theta\sec\phi, \text{ and } z=c\tan\phi.$$

This is seen at once if we substitute in the equation of the hyperboloid.

The two generating lines through the point P are the lines of intersection of the surface and the tangent plane at P. Now, the equation of the tangent plane at (θ, ϕ) is

$$\frac{x}{a}\cos\theta\sec\phi + \frac{y}{b}\sin\theta\sec\phi - \frac{z}{c}\tan\phi = 1;$$

hence the tangent plane meets the plane $z=0$ in the line whose equations are

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = \cos\phi,\ z=0 \ldots\ldots\ldots\text{(i)}.$$

If this line meet the section of the surface by $z=0$ in the points A, B, whose eccentric angles are α, β respectively, we have from (i)

$$\theta = \frac{\alpha+\beta}{2},\ \text{and}\ \phi = \frac{\alpha-\beta}{2}$$

or $$\alpha = \theta+\phi,\ \text{and}\ \beta = \theta - \phi \ \ldots\ldots\ldots\text{(ii)}.$$

Now AP, BP are the generators through P; hence from (ii), $\theta+\phi$ is constant for all points on the generator AP, and $\theta-\phi$ is constant for all points on the generator BP.

The direction-cosines of AP are proportional to

$$a(\cos\alpha - \cos\theta\sec\phi),\quad b(\sin\alpha - \sin\theta\sec\phi),\quad -c\tan\phi;$$

or proportional to

$$a\frac{\cos(\theta+\phi)\cos\phi-\cos\theta}{\sin\phi},\quad b\frac{\sin(\theta+\phi)\cos\phi-\sin\theta}{\sin\phi},\quad -c;$$

or to $$a\sin(\theta+\phi),\ -b\cos(\theta+\phi),\ c;$$

hence the equations of AP are

$$\frac{x-a\cos\theta\sec\phi}{a\sin(\theta+\phi)} = \frac{y-b\sin\theta\sec\phi}{-b\cos(\theta+\phi)} = \frac{z-c\tan\phi}{c}.$$

Similarly the equations of BP are

$$\frac{x-a\cos\theta\sec\phi}{a\sin(\theta-\phi)} = \frac{y-b\sin\theta\sec\phi}{-b\cos(\theta-\phi)} = \frac{z-c\tan\phi}{-c}.$$

COR. The equations of the generators, through the point on the principal elliptic section whose eccentric angle is θ, are

$$\frac{x - a\cos\theta}{a\sin\theta} = \frac{y - b\sin\theta}{-b\cos\theta} = \pm\frac{z}{c}.$$

These equations may also be obtained as follows:

The line whose equations are

$$\frac{x - a\cos\theta}{l} = \frac{y - b\sin\theta}{m} = \frac{z}{n} = r,$$

will meet the surface, where

$$\frac{(a\cos\theta + lr)^2}{a^2} + \frac{(b\sin\theta + mr)^2}{b^2} - \frac{n^2r^2}{c^2} = 1.$$

Hence, in order that the straight line may be a generating line, we must have

$$\frac{l^2}{a^2} + \frac{m^2}{b^2} - \frac{n^2}{c^2} = 0,$$

and
$$\frac{l\cos\theta}{a} + \frac{m\sin\theta}{b} = 0.$$

Whence
$$\frac{\frac{l}{a}}{\sin\theta} = \frac{\frac{m}{b}}{-\cos\theta} = \frac{\frac{n}{c}}{\pm 1}.$$

The equations of the generators are therefore

$$\frac{x - a\cos\theta}{a\sin\theta} = \frac{y - b\sin\theta}{-b\cos\theta} = \pm\frac{z}{c}.$$

139. *To find the equations of the generating lines through any point of a hyperbolic paraboloid.*

Let the equation of the paraboloid be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2z.$$

Let the equations of any line be

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r.$$

The points of intersection of the line and the surface are given by the equation

$$\frac{(\alpha+lr)^2}{a^2}-\frac{(\beta+mr)^2}{b^2}=2(\gamma+nr).$$

Hence, in order that the straight line may be a generating line, we must have

$$\frac{l^2}{a^2}-\frac{m^2}{b^2}=0 \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$

$$\frac{l\alpha}{a^2}-\frac{m\beta}{b^2}-n=0 \ldots\ldots\ldots\ldots\text{(ii)},$$

and

$$\frac{\alpha^2}{a^2}-\frac{\beta^2}{b^2}-2\gamma=0 \ldots\ldots\ldots\ldots\text{(iii)}.$$

The equation (iii) is satisfied if (α, β, γ) be any point on the surface; from (i) we have $\frac{l}{a}=\pm\frac{m}{b}$; and, substituting in (ii), we obtain

$$\frac{l}{a}=\frac{m}{\pm b}=\frac{n}{\frac{\alpha}{a}\mp\frac{\beta}{b}}.$$

Hence the equations of the two generating lines through the point (α, β, γ) are

$$\frac{x-\alpha}{a}=\frac{y-\beta}{\pm b}=\frac{z-\gamma}{\frac{\alpha}{a}\mp\frac{\beta}{b}} \ldots\ldots\ldots\ldots\ldots\ldots\text{(iv)}.$$

It is clear from the above that any generator of the paraboloid is parallel to one or other of the two planes

$$\frac{x}{a}\pm\frac{y}{b}=0.$$

Ex. 1. Shew that the projections of the generating lines of an hyperboloid on its principal planes are tangents to the principal sections.

The tangent plane at any point P on a principal section is perpendicular to that section. Hence the projection on the principal plane of any line in the tangent plane at P is the tangent line which is in the principal plane. This proves the proposition, since the generating lines through P are in the tangent plane at P.

Ex. 2. Find the locus of the point of intersection of perpendicular generators of an hyperboloid of one sheet.

If the generating lines at any point P are at right angles, the parallel central section is a rectangular hyperbola, and therefore the sum of the squares of its axes is zero. But the sum of the squares of three conjugate semi-diameters of the hyperboloid is constant and equal to $a^2+b^2-c^2$. Hence $OP^2=a^2+b^2-c^2$; so that the points are all on a sphere.

This is the result we should obtain by putting $\tan\theta=\infty$ in the result of Art. 136. We could also find the locus by using the equations of Art. 138.

Ex. 3. Find the angle between the generating lines at any point of a hyperbolic paraboloid.

The result is obtained at once from equations (iv), Art. 139. The generators are at right angles, if

$$a^2-b^2+\frac{\alpha^2}{a^2}-\frac{\beta^2}{b^2}=0, \text{ or if } 2\gamma+a^2-b^2=0.$$

Thus generators which are at right angles meet on the plane $z=\frac{1}{2}(b^2-a^2)$.

Ex. 4. A line moves so as always to intersect three given straight lines which are all parallel to the same plane: shew that it generates a hyperbolic paraboloid.

Ex. 5. A line moves so as always to intersect two given straight lines and to be parallel to a given plane: shew that it generates a hyperbolic paraboloid.

Ex. 6. AB and CD are two finite non-intersecting straight lines; shew that the lines which divide AB and CD in the same ratio are generators of one system of a hyperbolic paraboloid, and that the lines which divide AC and BD in the same ratio are generators of the opposite system of the same paraboloid.

Examples on Chapter VI.

1. A straight line revolves about a fixed straight line, find the surface generated. $\lambda^2 x^2 +$ [illegible] $-(\lambda^2-1)z^2=0$

2. If four non-intersecting straight lines be given, shew that the four hyperboloids which can be described, one through each set of three, all pass through two other straight lines.

3. Find the equation of the conicoid, three of whose generating lines are $x=0$, $y=a$; $y=0$, $z=a$; $z=0$, $x=a$. Shew that it is a surface of revolution, and find the eccentricity of its meridian section.

4. Find all the straight lines which can be drawn entirely coinciding (i) with the surface $y^3-z^3=3a^2x$; and (ii) with the surface $y^4-z^4=4a^3x$.

5. Normals are drawn to an hyperboloid of one sheet at every point through which the generators are at right angles; prove that the points, in which the normals intersect any one of the principal planes, lie in an ellipse.

6. Given any three lines, and a fourth line touching the hyperboloid through the three lines, then will each one of the four lines touch the hyperboloid through the other three lines.

7. A line is drawn through the centre of $ax^2+by^2+cz^2=1$ perpendicular to two parallel generators. Shew that such lines generate the cone

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=0.$$

8. If two generators of an hyperboloid be taken as two of the axes of co-ordinates shew that the equation of the surface is of the form

$$z^2+2fyz+2gzx+2hxy+2wz=0.$$

9. The generators through any point R on a ruled quadric intersect the generators at a fixed point O in P and Q. Shew that if the ratio $OP:OQ$ is constant, R lies on a plane section of the quadric which passes through O.

10. Find the locus of a point on an hyperboloid the generators through which intercept on two fixed generators portions whose product is constant.

11. If all the generators to an hyperboloid of one sheet be projected orthogonally on the tangent plane at any point, their envelope will be an hyperbola.

12. Find the equation of the locus of the foot of the perpendi-

cular from the point (a, O, O) on the different generating lines of the surface

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$$

13. Prove that the product of the sines of the angles that any generator makes with the planes of the circular sections is constant.

14. If CP, CD be conjugate semi-diameters of the principal elliptic section, and generators through P and D meet in T, prove that $TP^2=CD^2+c^2$, $TD^2=CP^2+c^2$.

15. If two generators drawn from O intersect the principal ellipse in points P, P', at the ends of conjugate diameters, then will

$$OP^2+OP'^2=a^2+b^2+2c^2.$$

16. The angle between the generating lines through the point (xyz) of the quadric $\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=1$ is $\cos^{-1}\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}$, where λ_1, λ_2, are the roots of the equation

$$\frac{x^2}{a(a+\lambda)}+\frac{y^2}{b(b+\lambda)}+\frac{z^2}{c(c+\lambda)}=0.$$

17. Shew that the shortest distances between generating lines of the same system drawn at the extremities of diameters of the principal elliptic section of the hyperboloid, whose equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,$$

lie on the surfaces whose equations are

$$\frac{cxy}{x^2+y^2}=\pm\frac{abz}{a^2-b^2}.$$

18. Prove that in general through two non-intersecting straight lines two and only two conicoids of revolution can be described.

19. The locus of points on $(abcfgh)(xyz)^2=1$ at which the generators are at right angles is the intersection of the surface with the sphere

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}(x^2+y^2+z^2)=bc+ca+ab-f^2-g^2-h^2.$$

20. Having given two generating lines that intersect and two points on an hyperboloid, shew that the locus of the centre is another hyperboloid bisecting the straight lines joining the two points to the intersection of the generators.

21. Shew that the volume of every parallelopiped which can be placed so that six of its edges lie along six of the generators of a given hyperboloid of one sheet is the same.

22. A solid hyperboloid has its generators marked on it and is then drawn in perspective: shew that the points of intersection of the representatives of consecutive generators of the same system will lie on an hyperbola.

23. If two points P, Q be taken on the surface

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-z^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)=1,$$

such that the tangent planes at those points are at right angles to one another, then will the two generating lines through P appear to be at right angles when seen from Q.

24. If two conicoids have a common generator, two of their common tangent planes through that generator have the same point of contact.

25. If AOA', BOB', COC' be any three straight lines, the lines AB, CA' $B'C'$ are generators of one system, and $A'B'$, $C'A$, BC are generators of the other system, of the same hyperboloid.

26. Deduce Pascal's Theorem from Dandelin's Theorem. [Ex. 6. Art. 135.]

27. If from any point on a hyperbolic paraboloid perpendiculars be let fall on all the generators of the surface of the same system, they will form a cone of the second degree.

28. If from any point on the surface of an hyperboloid of one sheet perpendiculars be drawn to all the generators of the same system, they will form a cone of the third degree.

29. The normals to a conicoid, at all points of a generating line, lie on a hyperbolic paraboloid.

30. In every rectilinear octagon $ABCDEFGH$ which is on a conicoid, the eight lines of intersection of the tangent planes at A, D; A, F; G, B; G, D; E, H; E, B; C, F; C, H are all generators of another conicoid. Also the lines AD, AF, GB, GD, HE, HC, CF, EB are all generators of another conicoid.

CHAPTER VII.

Systems of Conicoids. Tangential Equations. Reciprocation.

140. Since the general equation of the second degree contains nine constants, it follows that a conicoid will pass through any nine points, and that an infinite number of conicoids will pass through eight points.

If $S=0$, and $S'=0$ represent any two conicoids which pass through eight given points, then the equation $S+\lambda S'=0$ will be of the second degree, and will therefore represent a conicoid, and it is clear that the conicoid $S+\lambda S'=0$ will pass through all points common to $S=0$ and $S'=0$. Also, by giving a suitable value to λ, the conicoid $S+\lambda S'=0$ can be made to pass through any ninth point; and therefore will represent any conicoid through the eight given points.

Since the conicoid $S+\lambda S'=0$ not only passes through the eight given points, but also through all points on the curve of intersection of $S=0$ and $S'=0$, we see that *all conicoids through eight given points have a common curve of intersection.*

141. *Four cones will pass through the curve of intersection of two conicoids.*

Let the equations of any two conicoids be $F_1(x, y, z) = 0$ and $F_2(x, y, z) = 0$. The equation of any conicoid through their curve of intersection is of the form

$$F_1(x, y, z) + \lambda F_2(x, y, z) = 0.$$

The above equation will represent a cone, if

$$\begin{vmatrix} a_1 + \lambda a_2, & h_1 + \lambda h_2, & g_1 + \lambda g_2, & u_1 + \lambda u_2 \\ h_1 + \lambda h_2, & b_1 + \lambda b_2, & f_1 + \lambda f_2, & v_1 + \lambda v_2 \\ g_1 + \lambda g_2, & f_1 + \lambda f_2, & c_1 + \lambda c_2, & w_1 + \lambda w_2 \\ u_1 + \lambda u_2, & v_1 + \lambda v_2, & w_1 + \lambda w_2, & d_1 + \lambda d_2 \end{vmatrix} = 0.$$

Since the equation for determining λ is of the fourth degree, four cones, real or imaginary, will pass through the points of intersection of two conicoids.

142. *The vertices of the four cones through the curve of intersection of two conicoids are the angular points of a tetrahedron which is self-polar with respect to any conicoid which passes through that curve.*

Take the vertex O of one of the cones for origin, and let $F_1(x, y, z) = 0$ and $F_2(x, y, z) = 0$ be the equations of the two conicoids. Then the equation of the cone will be of the form $F_1(x, y, z) + \lambda F_2(x, y, z) = 0$. But, since the origin is at the vertex of the cone, its equation will be homogeneous. We therefore have

$$u_1 + \lambda u_2 = v_1 + \lambda v_2 = w_1 + \lambda w_2 = d_1 + \lambda d_2 = 0,$$

or $$\frac{u_1}{u_2} = \frac{v_1}{v_2} = \frac{w_1}{w_2} = \frac{d_1}{d_2} \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

Now the equation of the polar plane of O with respect to any conicoid

$$F_1(x, y, z) + \mu F_2(x, y, z) = 0, \text{ is}$$

$$(u_1 + \mu u_2)\, x + (v_1 + \mu v_2)\, y + (w_1 + \mu w_2)\, z + d_1 + \mu d_2 = 0;$$

and, from (i), it is clear that this polar plane coincides with

$$u_1 x + v_1 y + w_1 z + d_1 = 0$$

for all values of μ.

Hence O has the same polar plane with respect to all conicoids through the curve of intersection of the two given conicoids.

Now the polar plane of O with respect to any one of the other cones through the curve of intersection will pass through the vertex of that cone, and hence the vertices of the other three cones are on the polar plane of O with respect to any conicoid through the curve of intersection of the given conicoids: this proves the theorem.

143. If $S=0$ be the equation of any conicoid, and $\alpha\beta=0$ the equation of any two planes, then will $S-\lambda\alpha\beta=0$ be the general equation of a conicoid which passes through the two conics in which $S=0$ is cut by the planes $\alpha=0$ and $\beta=0$.

If now the plane $\alpha=0$ be supposed to move up to and ultimately coincide with the plane $\beta=0$, we obtain the form $S-\lambda\beta^2=0$, which represents a system of conicoids, all of which touch $S=0$ where it is met by the plane $\beta=0$.

The surfaces $S-\lambda\alpha\beta=0$ and $S=0$ touch one another at the two points where they are cut by the line whose equations are $\alpha=0$, $\beta=0$. For at either of these points the surfaces have *two* common tangent lines, namely the tangent lines to the sections by the planes $\alpha=0$ and $\beta=0$.

144. *All conicoids which pass through seven given points pass through another fixed point.*

Let $S_1=0$, $S_2=0$, $S_3=0$ be the equations of any three conicoids through the seven given points.

Then the conicoid whose equation is $S_1+\lambda S_2+\mu S_3=0$ will clearly pass through all points common to $S_1=0$, $S_2=0$ and $S_3=0$; and $S_1+\lambda S_2+\mu S_3=0$ can be made to coincide with any conicoid through the seven given points, for the two arbitrary constants λ and μ can be so chosen that the surface will pass through any two other points. Now the three conicoids $S_1=0$, $S_2=0$, $S_3=0$ have *eight* common points, all of which are on $S_1+\lambda S_2+\mu S_3=0$; this proves the theorem.

Thus, corresponding to any seven given points there is an eighth point associated with them, such that any conicoid through seven of the points will also pass through the eighth point; and it should be remarked that in order that a system of conicoids may have a common curve of intersection, they must have eight points in common which are not so associated.

Ex. 1. *All conicoids through the curve of intersection of two rectangular hyperboloids are rectangular hyperboloids.*

[A rectangular hyperboloid is one whose asymptotic cone has three perpendicular generating lines.]

The asymptotic cone of a conicoid has three generators at right angles when the sum of the coefficients of x^2, y^2 and z^2 in the equation of the surface is zero. Now the sum of the coefficients of x^2, y^2 and z^2 in $S+\lambda S'=0$ will be zero, if that sum is zero in S and also in S'. This proves the proposition.

Ex. 2. *Any two plane sections of a conicoid and the poles of those planes lie on another conicoid.*

Let $ax^2+by^2+cz^2+d=0$ be the conicoid, and let (x', y', z') and (x'', y'', z'') be any two points. The equations of the polar planes of these points will be $axx'+byy'+czz'+d=0$ and $axx''+byy''+czz''+d=0$.

The conicoid

$$\lambda(ax^2+by^2+cz^2+d)-(axx'+byy'+czz'+d)(axx''+byy''+czz''+d)=0$$

is the general equation of a conicoid through the two plane sections. The conicoid will pass through (x', y', z') if λ be such that

$$\lambda(ax'^2+by'^2+cz'^2+d)-(ax'^2+by'^2+cz'^2+d)(ax'x''+by'y''+cz'z''+d)=0,$$

or if $$\lambda=ax'x''+by'y''+cz'z''+d.$$

The symmetry of this result shews that the conicoid will likewise pass through (x'', y'', z'').

Ex. 3. *Through the curve of intersection of a sphere and an ellipsoid four quadric cones can be drawn; and if diameters of the ellipsoid be drawn parallel to the generators of one of the cones the diameters are all equal. Also the continued product of the four values of such diameters is equal to the continued product of the axes of the ellipsoid and of the diameter of the sphere.*

Let the equations of the ellipsoid and of the sphere be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

and $$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=r^2.$$

The general equation of a conicoid through the curve of intersection is

$$\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)+(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2-r^2=0\ldots\ldots\text{(i)}.$$

This conicoid will be a cone, if the co-ordinates of the centre satisfy the equations

$$\left(1+\frac{\lambda}{a^2}\right)x-\alpha=0,$$

$$\left(1+\frac{\lambda}{b^2}\right)y-\beta=0,$$

$$\left(1+\frac{\lambda}{c^2}\right)z-\gamma=0,$$

and

$$-\alpha x-\beta y-\gamma z+\alpha^2+\beta^2+\gamma^2-r^2-\lambda=0.$$

Eliminating x, y, z we have

$$\frac{a^2\alpha^2}{a^2+\lambda}+\frac{b^2\beta^2}{b^2+\lambda}+\frac{c^2\gamma^2}{c^2+\lambda}-\alpha^2-\beta^2-\gamma^2+r^2+\lambda=0\text{.........(ii)}.$$

If, for any particular value of λ, the conicoid given by (i) is a cone, the equation of the cone, when referred to its vertex, takes the form

$$\left(1+\frac{\lambda}{a^2}\right)x^2+\left(1+\frac{\lambda}{b^2}\right)y^2+\left(1+\frac{\lambda}{c^2}\right)z^2=0;$$

and therefore the direction-cosines of any diameter which is parallel to one of the generating lines of the cone, satisfy the equation

$$\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}=-\frac{1}{\lambda}.$$

Hence the square of the semi-diameter is constant and equal to $-\lambda$.

Hence also the continued product of the squares of the four values of the semi-diameters is equal to the product of the four roots of the equation (ii); and the product of the roots is easily seen to be $a^2b^2c^2r^2$.

Ex. 4. *The locus of the centres of all conicoids which pass through seven given points is a cubic surface, which passes through the middle point of the line joining any pair of the seven given points.*

Let $S_1=0$, $S_2=0$, $S_3=0$ be any three conicoids through the seven given points; then the general equation of the conicoids is

$$S_1+\lambda S_2+\mu S_3=0.$$

The equations for the centre are

$$\frac{dS_1}{dx}+\lambda\frac{dS_2}{dx}+\mu\frac{dS_3}{dx}=0,$$

$$\frac{dS_1}{dy}+\lambda\frac{dS_2}{dy}+\mu\frac{dS_3}{dy}=0,$$

$$\frac{dS_1}{dz}+\lambda\frac{dS_2}{dz}+\mu\frac{dS_3}{dz}=0.$$

Hence the equation of the locus of the centres, for different values of λ and μ, is

$$\begin{vmatrix} \frac{dS_1}{dx}, & \frac{dS_2}{dx}, & \frac{dS_3}{dx} \\ \frac{dS_1}{dy}, & \frac{dS_2}{dy}, & \frac{dS_3}{dy} \\ \frac{dS_1}{dz}, & \frac{dS_2}{dz}, & \frac{dS_3}{dz} \end{vmatrix} = 0,$$

which is a cubic surface, since $\frac{dS_1}{dx}$ &c. are of the first degree.

Now, to have the centre of a conicoid given, is equivalent to having *three* conditions given; hence a conicoid which has a given centre can be made to pass through any *six* points. Hence, if A, B be any two of the seven given points, one conicoid whose centre is the middle point of AB will pass through A and through the remaining five points; and a conicoid whose centre is the middle point of AB, and which goes through A, must also go through B. Thus the middle point of AB is a point on the locus of centres; and so also is the middle point of the line joining any other pair of the given points.

[*Messenger of Mathematics*, vol. XIII. p. 145, and XIV. p. 97.]

Tangential Equations.

145. If the equation of a plane be $lx + my + nz + 1 = 0$, then the position of the plane is determined if l, m, n are known, and by changing the values of l, m and n the equation may be made to represent any plane whatever. The quantities l, m, and n which thus define the position of a plane are called the *co-ordinates* of the *plane*. These co-ordinates, when their signs are changed, are the reciprocals of the intercepts on the axes.

If the co-ordinates of a plane be connected by any relation, the plane will envelope a surface; and the equation which expresses the relation is called the *tangential equation* of the surface.

146. If the tangential equation of a surface be of the n^{th} degree, then n tangent planes can be drawn to the surface through any straight line. For, let the straight line be given by the equations $ax + by + cz + 1 = 0$, $a'x + b'y + c'z + 1 = 0$; then the co-ordinates of any plane through the line will be $\frac{a + \lambda a'}{1 + \lambda}$, $\frac{b + \lambda b'}{1 + \lambda}$ and $\frac{c + \lambda c'}{1 + \lambda}$. If these co-ordinates be sub-

stituted in the given tangential equation, we shall obtain an equation of the n^{th} degree for the determination of λ, which proves the proposition.

Def. A surface is said to be of the n^{th} *class* when n tangent planes can be drawn to it through an arbitrary straight line.

147. We have shewn in Art. 57 that the plane

$$lx + my + nz + 1 = 0$$

will touch the conicoid whose equation is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0,$$

if
$$Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm$$
$$+ 2Ul + 2Vm + 2Wn + D = 0,$$

where A, B, C... are the co-factors of a, b, c... in the discriminant.

Hence the tangential equation of a conicoid is of the second degree.

Conversely every surface whose tangential equation is of the second degree is a conicoid.

148. Since the tangential equation of a conicoid is of the second degree, which in its most general form contains nine constants, it follows that a conicoid can be made to satisfy nine conditions and no more; and in particular a conicoid can be made to touch nine given planes.

149. *To find the Cartesian co-ordinates of the centre of the conicoid given by the general tangential equation of the second degree.*

The two tangent planes to the conicoid which are parallel to the plane $x = 0$ are those for which $m = n = 0$. The values of l are therefore given by the equation $al^2 + 2ul + d = 0$.

Now the centre of the surface is on the plane midway between these; and hence the centre is on the plane $x = \frac{u}{d}$.

Similarly the centre is on the planes $y=\frac{v}{d}$, and $z=\frac{w}{d}$.

Hence the required co-ordinates are $\frac{u}{d}, \frac{v}{d}, \frac{w}{d}$. [See Art. 76.]

150. We may take the equation of the moving plane to be $lx+my+nz+p=0$; and the plane will envelope a surface if l, m, n, p be connected by a *homogeneous* equation; for any homogeneous equation in l, m, n, p would be equivalent to an equation between the constants $\frac{l}{p}, \frac{m}{p}, \frac{n}{p}$.

If we take $lx+my+nz+p=0$ for the equation of the plane, we may suppose l, m, n to be the direction-cosines of the normal to the plane.

151. *To find the director-sphere of a conicoid whose tangential equation is given.*

If we eliminate p between the equation of the surface and the equation $lx+my+nz+p=0$, we shall obtain a relation between the direction-cosines of any tangent plane which passes through the particular point (x, y, z). The relation will be

$$al^2+bm^2+cn^2+d(lx+my+nz)^2+2fmn+2gnl+2hlm \\ -2(ul+vm+wn)(lx+my+nz)=0.$$

If (x, y, z) be a point on the director-sphere, three perpendicular tangent planes will pass through it; the above relation must therefore be satisfied by the direction-cosines of each of three perpendicular planes. Hence, by addition, we have

$$a+b+c-2ux-2vy-2wz+d(x^2+y^2+z^2)=0,$$

which is the required equation of the director-sphere.

152. If $S=0$ and $S'=0$ be the tangential equations of any two conicoids which touch eight given planes, then the equation $S+\lambda S'=0$ will be of the second degree, and will therefore be the tangential equation of a conicoid; and it is clear that the conicoid $S+\lambda S'=0$ will touch the common

tangent planes of $S=0$ and $S'=0$, for if the co-ordinates of any plane satisfy the equations $S=0$ and $S'=0$, they will also satisfy the equation $S+\lambda S'=0$. Also, by giving a suitable value to λ, the conicoid $S+\lambda S'=0$ can be made to touch any ninth plane: it will therefore represent any conicoid touching the eight given planes.

153. If $S_1=0$, $S_2=0$, $S_3=0$ be the tangential equations of any three conicoids which touch seven given planes; then the conicoid whose tangential equation is $S_1+\lambda S_2+\mu S_3=0$ will touch each of the seven given planes, for if the co-ordinates of any plane satisfy the three equations $S_1=0$, $S_2=0$ and $S_3=0$, it will also satisfy the equation

$$S_1+\lambda S_2+\mu S_3=0.$$

Also, by giving suitable values to λ and μ, the conicoid

$$S_1+\lambda S_2+\mu S_3=0$$

can be made to touch any *two* other planes; hence

$$S_1+\lambda S_2+\mu S_3=0$$

is the most general equation of a conicoid which touches the seven given planes.

Similarly, if $S_1=0$, $S_2=0$, $S_3=0$ and $S_4=0$ be the tangential equations of any four conicoids which touch six given planes, $S_1+\lambda S_2+\mu S_3+\nu S_4=0$ will be the general tangential equation of the conicoids which touch those six planes.

Ex. 1. *The centres of all conicoids which touch eight given planes are on a straight line.*

If $S=0$ and $S'=0$ be the equations of any two conicoids which touch the eight given planes, then $S+\lambda S'=0$ will be the general equation of a conicoid touching them. The centre of the conicoid is given by

$$x=\frac{u+\lambda u'}{d+\lambda d'},\ y=\frac{v+\lambda v'}{d+\lambda d'},\ z=\frac{w+\lambda w'}{d+\lambda d'}.$$

Eliminating λ we obtain the equation of the centre locus, namely

$$\frac{dx-u}{d'x-u'}=\frac{dy-v}{d'y-v'}=\frac{dz-w}{d'z-w'};$$

hence the locus is a straight line.

Ex. 2. *The centres of all conicoids which touch seven given planes are on a plane.*

If $S=0$, $S'=0$, $S''=0$ be the equations of three conicoids which touch the seven given planes, then the general equation of a conicoid which touches the planes will be
$$S+\lambda S'+\mu S''=0.$$

Ex. 3. *The director-spheres of all conicoids which have eight common tangent planes have a common radical plane.*

The director-sphere of the conicoid $S+\lambda S'=0$ is
$$a+b+c-2ux-2vy-2wz+d(x^2+y^2+z^2)$$
$$+\lambda\{a'+b'+c'-2u'x-2v'y-2w'z+d'(x^2+y^2+z^2)\}=0.$$

Ex. 4. *The director-spheres of all conicoids which touch six given planes are cut orthogonally by the same sphere.* [P. Serret's Theorem.]

If $C_1=0$, $C_2=0$, $C_3=0$ and $C_4=0$ be the equations of any four conicoids which touch the six planes; then the general equation of the conicoids will be
$$C_1+\lambda C_2+\mu C_3+\nu C_4=0.$$

Now from Art. 151 we see that the equation of the director-sphere of a conicoid is linear in a, b, c, &c. It therefore follows that, if $S_1=0$, $S_2=0$, $S_3=0$ and $S_4=0$ be the equations of the director-spheres of the conicoids $C_1=0$, $C_2=0$, $C_3=0$ and $C_4=0$ respectively, the equation of the director-sphere of
$$C_1+\lambda C_2+\mu C_3+\nu C_4=0$$
will be
$$S_1+\lambda S_2+\mu S_3+\nu S_4=0.$$

Now from the condition that two spheres may cut orthogonally [Art. 90, Ex. 6], it follows that a sphere can always be formed which will cut four given spheres orthogonally; and it also follows that the sphere which cuts orthogonally the four spheres $S_1=0$, $S_2=0$, $S_3=0$ and $S_4=0$, will cut orthogonally any sphere whose equation is $S_1+\lambda S_2+\mu S_3+\nu S_4=0$. This proves the proposition.

Ex. 5. *The locus of the centres of conicoids which touch six planes, and have the sum of the squares of their axes given, is a sphere.* [Mention's Theorem.]

By Ex. 4 all the director-spheres of the conicoids are cut orthogonally by the same sphere; and the director-spheres have a constant radius. Hence their centres, which are the centres of the conicoids, are on a sphere concentric with this orthogonal sphere.

Reciprocation.

154. If we have any system of points and planes in space, and we take the polar planes of those points and the poles of the planes, with respect to a fixed conicoid C, we obtain another system of planes and points which is called

the *polar reciprocal* of the former with respect to the *auxiliary* conicoid C.

When a point in one system and a plane in the reciprocal are pole and polar plane with respect to the auxiliary conicoid C, we shall say that they correspond to one another.

If in one system we have a surface S, the planes which correspond to the different points of S will all touch some surface S'. Let the planes corresponding to any number of points P, Q, R... on a plane section of S meet in T; then T is the pole of the plane PQR with respect to C, that is the plane PQR corresponds to T. Now, if the plane PQR move up to and ultimately coincide with the tangent plane at P, the corresponding tangent planes to S' will ultimately coincide with one another, and their point of intersection T will ultimately be on the surface S'. So that a tangent plane to the surface S corresponds to a point on the surface S', just as a tangent plane to S' corresponds to a point on S. Hence we see that S is generated from S' exactly as S' is from S.

155. To a line L in one system corresponds the line L' in the reciprocal system which is the polar line of L with respect to the auxiliary conicoid.

If any line L cut the surface S in any number of points P, Q, R... we shall have tangent planes to S' corresponding to the points P, Q, R..., and these tangent planes will all pass through a line, viz. through the polar line of L with respect to the auxiliary conicoid. Hence, as many tangent planes to S' can be drawn through a straight line as there are points on S lying on a straight line. That is to say the *class* [Art. 146] of S' is equal to the degree of S. Reciprocally the degree of S' is equal to the class of S.

In particular, if S be a conicoid it is of the second degree and of the second class; hence S' is of the second class and of the second degree, and is therefore also a conicoid.

156. The reciprocal of a point which is common to two surfaces is a plane which touches both the reciprocal surfaces.

If two surfaces have a common curve of intersection, they have an infinite number of common points; the reciprocal surfaces therefore have an infinite number of common tangent planes. These common tangent planes form a surface: and, since the line of intersection of any two consecutive planes is on the surface, it is a *ruled* surface, the generating lines being the lines of intersection of consecutive planes. Any one of the planes contains two consecutive generating lines, so that two consecutive generators must intersect; hence the surface is a *developable* surface.

If all the points of the curve lie on a plane, all the tangent planes to the developable pass through a point; the developable must therefore be a cone. Hence the reciprocal of a plane curve is a cone.

It follows by reciprocation from Art. 144, that all conicoids which touch seven fixed planes will touch an associated eighth plane.

It also follows from Art. 140 that all conicoids which touch eight given planes have an infinite number of common tangent planes, provided that the eight given planes do not form an associated system.

157. The reciprocation is usually taken with respect to a sphere, and since the nature of the reciprocal surface is independent of the radius of the sphere, we only require to know the centre of the sphere, which is called the origin of reciprocation.

The line joining the centre of a sphere to any point is perpendicular to the polar plane of the point. Hence, if P, Q be any two points, the angle between the polar planes of these points with respect to a sphere is equal to the angle that PQ subtends at the centre of the sphere.

158. If any conicoid be reciprocated with respect to a point O, the points on the reciprocal surface which correspond to the tangent planes through O to the original surface must be at an infinite distance.

Hence the generating lines of the asymptotic cone of the reciprocal surface are perpendicular to the tangent planes of the enveloping cone from O to the original surface.

In particular, if the point O be on the director-sphere of the original surface, that is if three of the tangent planes from O be at right angles, the asymptotic cone of the reciprocal surface will have three generating lines at right angles.

Corresponding to a point at infinity on the original surface we have a tangent plane through O to the reciprocal surface.

Hence the tangent cone from the origin to the reciprocal surface has its tangent planes perpendicular to the generating lines of the asymptotic cone of the original surface.

In particular, if the asymptotic cone of the original surface have three perpendicular generating lines, three of the tangent planes from O to the reciprocal surface will be at right angles, so that O is a point on the director-sphere of the reciprocal conicoid.

159. As an example of reciprocation take the theorem:— "If two of the conicoids which pass through eight given points are rectangular hyperboloids, they will all be rectangular hyperboloids." If this be reciprocated with respect to any point O we obtain the following, "If the director-spheres of two of the conicoids which touch eight given planes pass through a point O, the director-spheres of all the conicoids will pass through O." Hence "the director-spheres of all conicoids which touch eight given planes have a common radical plane."

As another example of reciprocation take the theorem:— "A straight line is drawn to cut the faces of a tetrahedron $ABCD$ which are opposite to the angles A, B, C, D in a, b, c and d respectively. Shew that the spheres described on the straight lines Aa, Bb, Cc, and Dd as diameters have a common radical axis."

Let O be a point of intersection of the spheres whose diameters are Aa, Bb and Cc. If we reciprocate with respect to O we shall obtain another tetrahedron whose faces and angular points correspond respectively to the angular points and faces of the original tetrahedron. Corresponding to the four points a, b, c, d which are on a straight line, we shall have four planes with a common line of intersection; and, since a, b, c, d are on the faces of the original tetrahedron, the corresponding planes will pass through the angular points of the reciprocal tetrahedron; also since the angles AOa, BOb, COc are right angles, the three pairs of planes corresponding respectively to a and A, to b and B, and to c and C will be at right angles; this shews that the line of intersection of the planes corresponding to a, b, c, d will meet three of the perpendiculars of the reciprocal tetrahedron. But we know [Art. 135, Ex. 4], that every line which meets three of the perpendiculars of a tetrahedron, meets the remaining perpendicular; and hence the planes corresponding to d and D are at right angles, which shews that the angle dOD is a right angle. Hence O is also on the sphere whose diameter is Dd.

Ex. 1. The reciprocal of a sphere with respect to any point is a conicoid of revolution.

Ex. 2. Find the reciprocal of $ax^2+by^2+cz^2=1$ with respect to the sphere $x^2+y^2+z^2=1$. *Ans.* $\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=1.$

Ex. 3. Shew that the reciprocal of a ruled surface is a ruled surface.

Ex. 4. Shew that if two conicoids have one common enveloping cone they also have another. [The reciprocal of Art. 120.]

Ex. 5. Either of the two surfaces $ax^2+by^2=\pm 2z$ is self reciprocal with respect to the other.

Examples on Chapter VII.

1. When three conicoids pass through the same conic, the planes of their other conics of intersection pass through the same line.

2. Shew that, if the curve of intersection of two conicoids cross itself, the conicoids will touch at the point of crossing; and that if the curve of intersection cross itself twice, it will consist of two conics.

3. Shew that three paraboloids will pass through the curve of intersection of any two conicoids.

4. Shew that a surface of revolution will go through the intersection of any two conicoids whose axes are parallel.

5. If a conicoid have double contact with a sphere, the square of the tangent to the sphere from any point on the conicoid is in a constant ratio to the product of the distances of that point from the planes of intersection.

6. Any two conicoids which have a common enveloping cone intersect in plane curves.

7. Shew that the polar lines of a fixed line, with respect to a system of conicoids through eight given points, generate an hyperboloid of one sheet.

8. Shew that the polar planes of a fixed point, with respect to a system of conicoids through seven given points, pass through a fixed point.

9. Shew that the poles of a fixed plane, with respect to a system of conicoids which touch seven given planes, lie on a fixed plane.

10. The polar planes of a point with respect to two given conicoids are at right angles; shew that the locus of the point is another conicoid.

11. All conicoids through the intersection of a sphere and a given conicoid, have their principal planes, and also their cyclic planes, in fixed directions.

12. If O be any point on a conicoid, and lines be drawn through O parallel to equal diameters of the conicoid, these lines will meet the surface on a sphere whose centre is on the normal at O.

13. If O be the centre of any conicoid through the intersection of a sphere and a given conicoid, the line joining O to the centre of the sphere is perpendicular to the polar plane of O with respect to the given conicoid.

14. Shew that, in a system of conicoids which have a common curve of intersection, the diametral planes of parallel diameters have a common line of intersection.

15. If a system of conicoids be drawn through the intersection of a given conicoid and a sphere whose centre is O, the normals to them from O form a cone of the second degree, and their feet are on a curve of the third order which is the locus of the centres of all the surfaces.

16. If any point on a given diameter of an ellipsoid be joined to every point of a given plane section of the surface, the cone so formed will meet the surface in another plane section, whose envelope will be a hyperbolic cylinder.

17. A cone is described with its vertex at a fixed point, and one axis parallel to an axis of a given quadric, and the cone cuts the quadric in plane curves; shew that these planes envelope a parabolic cylinder whose directrix-plane passes through the fixed point.

18. If two spheres be inscribed in any conicoid of revolution, any common tangent plane of the spheres will cut the conicoid in a conic having its points of contact for foci.

19. If the line joining the point of intersection of three, out of six given planes, to the point of intersection of the other three, be called a diagonal; shew that the ten spheres described on the diagonals have the same radical centre, and the same orthogonal sphere.

20. The circumscribing sphere of a tetrahedron which is self polar with respect to a conicoid cuts the director-sphere of the conicoid orthogonally.

CHAPTER VIII.

Confocal Conicoids. Concyclic Conicoids. Foci of Conicoids.

160. Conicoids whose principal sections are confocal conics are called confocal conicoids.

The general equation of a system of confocal conicoids is

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1.$$

Suppose a, b, c to be in descending order of magnitude.

If λ is positive, the surface is an ellipsoid, and the principal axes of the surface will increase as λ increases, and their ratio will tend more and more to equality as λ is increased more and more; so that a sphere of infinite radius is a limiting form of one of the confocals.

If λ is negative and less than c^2 the surface is an ellipsoid; but the ellipsoid becomes flatter and flatter as λ approaches the value $-c^2$. Hence the elliptic disc whose equations are

$$z=0,\quad \frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2}=1,$$

is a limiting form of one of the confocals.

If λ is between $-c^2$ and $-b^2$ the surface is an hyperboloid of one sheet. When λ is very nearly equal to $-c^2$, the hyperboloid is very nearly coincident with that part of the plane $z=0$ which is exterior to the ellipse $\dfrac{x^2}{a^2-c^2}+\dfrac{y^2}{b^2-c^2}=1.$

When λ is very nearly equal to $-b^2$, the hyperboloid is very nearly coincident with that part of the plane $y=0$ which contains the centre and is bounded by the hyperbola

$$\frac{x^2}{a^2-b^2}+\frac{z^2}{c^2-b^2}=1.$$

If λ is between $-b^2$ and $-a^2$, the surface is an hyperboloid of two sheets. When λ is very nearly equal to $-b^2$, the hyperboloid is very nearly coincident with that part of the plane $y=0$ which does not contain the centre and is bounded by the hyperbola $\dfrac{x^2}{a^2-b^2}+\dfrac{z^2}{c^2-b^2}=1$.

When λ is between $-a^2$ and $-\infty$ the surface is imaginary. The two conics

$$z=0,\ \frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2}=1,$$

and

$$y=0,\ \frac{x^2}{a^2-b^2}+\frac{z^2}{c^2-b^2}=1,$$

which we have seen are the boundaries of limiting forms of confocal conicoids, are called *focal conics*, one being the focal ellipse, and the other the focal hyperbola.

161. *Three conicoids, confocal with a given central conicoid, will pass through a given point; and one of the three is an ellipsoid, one an hyperboloid of one sheet, and one an hyperboloid of two sheets.*

Let the equation of the given conicoid be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Any conicoid confocal to this is

$$\frac{x^2}{a^2-\lambda}+\frac{y^2}{b^2-\lambda}+\frac{z^2}{c^2-\lambda}=1\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

This will pass through the particular point (f, g, h) if

$$f^2(b^2-\lambda)(c^2-\lambda)+g^2(c^2-\lambda)(a^2-\lambda)$$
$$+h^2(a^2-\lambda)(b^2-\lambda)-(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)=0\ldots\ldots\text{(ii)}.$$

If we substitute for λ the values a^2, b^2, c^2, and $-\infty$ in succession, the left side of the equation (ii) will be $+, -, +, -$; hence there are three real roots of the equation, namely one between a^2 and b^2, one between b^2 and c^2, and one between c^2 and $-\infty$. When λ is between c^2 and $-\infty$, all the coefficients in (i) are positive, and the surface is an ellipsoid; when λ is between c^2 and b^2, one of the coefficients is negative, and the surface is an hyperboloid of one sheet; and when λ is between b^2 and a^2 two of the coefficients are negative, and the surface is an hyperboloid of two sheets.

162. *One conicoid of a given confocal system will touch any plane.*

Let the equation of the plane be

$$lx + my + nz = p.$$

The plane will touch the conicoid

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1,$$

if $$(a^2+\lambda)\,l^2 + (b^2+\lambda)\,m^2 + (c^2+\lambda)\,n^2 = p^2,$$

which gives one, and only one, value of λ. Hence one confocal will touch the given plane.

163. *Two conicoids of a confocal system will touch any straight line.*

Let the straight line be the line of intersection of the planes $lx + my + nz + p = 0$, $l'x + m'y + n'z + p' = 0$. Any plane through the straight line will be

$$(l + kl')\,x + (m + km')\,y + (n + kn')\,z + (p + kp') = 0.$$

This plane will touch the conicoid

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1,$$

if $$(a^2+\lambda)(l+kl')^2 + (b^2+\lambda)\,(m+km')^2 + (c^2+\lambda)\,(n+kn')^2 = (p+kp')^2.$$

Now, if the given line be a tangent line of the conicoid, the two tangent planes through it will coincide. Hence the roots of the above equation in k must be equal. The condition for this gives the following equation for finding λ,

$$\{(a^2+\lambda)\,l^2+(b^2+\lambda)\,m^2+(c^2+\lambda)\,n^2-p^2\}$$
$$\{(a^2+\lambda)\,l'^2+(b^2+\lambda)\,m'^2+(c^2+\lambda)\,n'^2-p'^2\}$$
$$=\{(a^2+\lambda)\,ll'+(b^2+\lambda)\,mm'+(c^2+\lambda)\,nn'-pp'\}^2.$$

Since the equation is of the second degree, there are *two* confocals which touch the given line.

164. *Two confocal conicoids cut one another at right angles at all their common points.*

Let the equations of the conicoids be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1,$$

and let $(x'y'z')$ be a common point; then the co-ordinates x', y', z' will satisfy both the above equations. Hence, by subtraction we have

$$\frac{x'^2}{a^2(a^2+\lambda)}+\frac{y'^2}{b^2(b^2+\lambda)}+\frac{z'^2}{c^2(c^2+\lambda)}=0 \;\ldots\ldots\text{(i)}.$$

Now the equations of the tangent planes at the common point $(x'y'z')$ are

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}+\frac{zz'}{c^2}=1,$$

and $$\frac{xx'}{a^2+\lambda}+\frac{yy'}{b^2+\lambda}+\frac{zz'}{c^2+\lambda}=1, \text{ respectively.}$$

The condition (i) shews that these tangent planes are at right angles.

165. *If a straight line touch two confocal conicoids, the tangent planes at the points of contact will be at right angles.*

Let $(x'y'z')$, $(x''y''z'')$ be the points of contact, and let the conicoids be

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1,$$

and
$$\frac{x^2}{a^2+\lambda'}+\frac{y^2}{b^2+\lambda'}+\frac{z^2}{c^2+\lambda'}=1.$$

The tangent planes will be at right angles if

$$\frac{x'x''}{(a^2+\lambda)(a^2+\lambda')}+\frac{y'y''}{(b^2+\lambda)(b^2+\lambda')}+\frac{z'z''}{(c^2+\lambda)(c^2+\lambda')}=0 \ldots\text{(i)}.$$

But, since the line joining the two points is a tangent line to both conicoids, each point must be in the tangent plane at the other. Hence

$$\frac{x'x''}{a^2+\lambda}+\frac{y'y''}{b^2+\lambda}+\frac{z'z''}{c^2+\lambda}=1,$$

and
$$\frac{x'x''}{a^2+\lambda'}+\frac{y'y''}{b^2+\lambda'}+\frac{z'z''}{c^2+\lambda'}=1.$$

By subtraction we see that the condition (i) is satisfied.

Ex. 1. The difference of the squares of the perpendiculars from the centre on any two parallel tangent planes to two given confocal conicoids is constant. [$p_1^2-p_2^2=\lambda_1-\lambda_2$.]

Ex. 2. The locus of the point of intersection of three planes mutually at right angles, each of which touches one of three given confocals, is a sphere. [See Art. 92.]

Ex. 3. The locus of the umbilici of a system of confocal ellipsoids is the focal hyperbola.

[The umbilici are given by

$$\frac{x}{\sqrt{(a^2+\lambda)}}=\pm\sqrt{\frac{a^2-b^2}{a^2-c^2}},\quad y=0,\quad \frac{z}{\sqrt{(c^2+\lambda)}}=\pm\sqrt{\frac{b^2-c^2}{a^2-c^2}}.]$$

Ex. 4. If two concentric and co-axial conicoids cut one another everywhere at right angles they must be confocal.

Ex. 5. P, Q are two points, one on each of two confocal conicoids, and the tangent planes at P, Q meet in the line RS; shew that, if the plane through RS and the centre bisect the line PQ, the tangent planes at P and Q must be at right angles to one another.

Ex. 6. Shew that two confocal paraboloids cut everywhere at right angles. [The general equation of confocal paraboloids is $\frac{x^2}{l+\lambda}+\frac{y^2}{m+\lambda}=2z+\lambda$.]

166. We have seen that three conicoids confocal with a given conicoid will pass through any point P, the parameters of the confocals being the three values of λ given by the equation

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1,$$

where x, y, z are the co-ordinates of P.

If the roots of the above equation be λ_1, λ_2, λ_3, it is easy to shew that

$$x^2=\frac{(a^2+\lambda_1)(a^2+\lambda_2)(a^2+\lambda_3)}{(a^2-b^2)(a^2-c^2)},$$

with similar values for y^2 and z^2.

Hence the absolute values of the co-ordinates of any point can be expressed in terms of the parameters of the conicoids which meet in that point, and are confocal with a given conicoid.

167. *The parameters of the two confocals through any point P of a conicoid are equal to the squares of the axes of the central section of the conicoid which is parallel to the tangent plane at P; and the normals at P to the confocals are parallel to the axes of that section.*

Let (x', y', z') be any point P on the conicoid whose equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1;$$

then, if P be on the confocal whose parameter is λ, we have

$$\frac{x'^2}{a^2-\lambda}+\frac{y'^2}{b^2-\lambda}+\frac{z'^2}{c^2-\lambda}=1;$$

and therefore

$$\frac{x'^2}{a^2(a^2-\lambda)}+\frac{y'^2}{b^2(b^2-\lambda)}+\frac{z'^2}{c^2(c^2-\lambda)}=0 \dots\dots\dots \text{(i)}.$$

The equation of the central section parallel to the tangent plane at P is

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}+\frac{zz'}{c^2}=0.$$

Hence the equation giving the squares of the axes of the section is

$$\frac{\dfrac{x'^2}{a^4}}{\dfrac{1}{a^2}-\dfrac{1}{r^2}}+\frac{\dfrac{y'^2}{b^4}}{\dfrac{1}{b^2}-\dfrac{1}{r^2}}+\frac{\dfrac{z'^2}{c^4}}{\dfrac{1}{c^2}-\dfrac{1}{r^2}}=0 \text{ [Art. 115]},$$

or $$\frac{x'^2}{a^2(a^2-r^2)}+\frac{y'^2}{b^2(b^2-r^2)}+\frac{z'^2}{c^2(c^2-r^2)}=0\ldots\ldots \text{(ii)}.$$

Comparing (i) and (ii), we see that the squares of the axes of the section are the two values of λ.

The equations of the diameter which is parallel to the normal at P to one of the confocals are

$$\frac{x}{\dfrac{x'}{a^2-\lambda}}=\frac{y}{\dfrac{y'}{b^2-\lambda}}=\frac{z}{\dfrac{z'}{c^2-\lambda}}.$$

The length of the diameter will be equal to $2\sqrt{\lambda}$ if it be one of the generating lines of the cone

$$x^2\left(\frac{1}{a^2}-\frac{1}{\lambda}\right)+y^2\left(\frac{1}{b^2}-\frac{1}{\lambda}\right)+z^2\left(\frac{1}{c^2}-\frac{1}{\lambda}\right)=0 \text{ [Art. 73, Ex. 5]};$$

the condition that this may be the case is

$$\frac{x'^2}{(a^2-\lambda)^2}\left(\frac{1}{a^2}-\frac{1}{\lambda}\right)+\frac{y'^2}{(b^2-\lambda)^2}\left(\frac{1}{b^2}-\frac{1}{\lambda}\right)+\frac{z'^2}{(c^2-\lambda)^2}\left(\frac{1}{c^2}-\frac{1}{\lambda}\right)=0;$$

and it is clear from (i) that this condition is satisfied.

Hence an axis of the central section is parallel to the normal to one of the confocals through P, and the square of the length of the semi-axis is equal to the parameter of that confocal.

COR. If diameters of a conicoid be drawn parallel to the normals to a confocal at all points of their curve of intersection, such diameters will be of constant length.

168. Two points (x, y, z), (ξ, η, ζ), one on each of two co-axial conicoids whose equations are

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=1,$$

respectively, are said to correspond when

$$\frac{x}{a}=\frac{\xi}{\alpha},\ \frac{y}{b}=\frac{\eta}{\beta} \text{ and } \frac{z}{c}=\frac{\zeta}{\gamma}.$$

In order that real points on one conicoid may correspond to real points on the other, the two surfaces must be of the same nature, and must be similarly placed.

It follows at once from the equations (i), Art. 96, that if on one of the conicoids three points be taken which are extremities of conjugate diameters, the three corresponding points on the other conicoid will be at extremities of conjugate diameters.

169. *The distance between two points, one on each of two confocal ellipsoids, is equal to the distance between the two corresponding points.*

Let (x_1, y_1, z_1), (x_2, y_2, z_2) be the two points on one conicoid, and (ξ_1, η_1, ζ_1), (ξ_2, η_2, ζ_2) the corresponding points on the other conicoid.

Then $$\frac{x_1}{a}=\frac{\xi_1}{\alpha},\ \frac{y_1}{b}=\frac{\eta_1}{\beta},\ \frac{z_1}{c}=\frac{\zeta_1}{\gamma};$$

and $$\frac{x_2}{a}=\frac{\xi_2}{\alpha},\ \frac{y_2}{b}=\frac{\eta_2}{\beta},\ \frac{z_2}{c}=\frac{\zeta_2}{\gamma}.$$

We have to prove that

$$(x_1-\xi_2)^2+(y_1-\eta_2)^2+(z_1-\zeta_2)^2=(x_2-\xi_1)^2+(y_2-\eta_1)^2+(z_2-\zeta_1)^2,$$

or $$\left(\frac{a}{\alpha}\xi_1-\frac{\alpha}{a}x_2\right)^2+\left(\frac{b}{\beta}\eta_1-\frac{\beta}{b}y_2\right)^2+\left(\frac{c}{\gamma}\zeta_1-\frac{\gamma}{c}z_2\right)^2$$

$$=(x_2-\xi_1)^2+(y_2-\eta_1)^2+(z_2-\zeta_1)^2,$$

or $$(a^2-\alpha^2)\left(\frac{\xi_1^2}{\alpha^2}-\frac{x_2^2}{a^2}\right)+(b^2-\beta^2)\left(\frac{\eta_1^2}{\beta^2}-\frac{y_2^2}{b^2}\right)$$
$$+(c^2-\gamma^2)\left(\frac{\zeta_1^2}{\gamma^2}-\frac{z_2^2}{c^2}\right)=0,$$

which is clearly the case, since the conicoids are confocal, and
$$\frac{\xi_1^2}{\alpha^2}+\frac{\eta_1^2}{\beta^2}+\frac{\zeta_1^2}{\gamma^2}=\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}+\frac{z_2^2}{c^2}=1.$$

170. *The locus of the poles of a given plane with respect to a system of confocal conicoids is a straight line.*

Let the equation of the confocals be
$$\frac{x^2}{a^2-\lambda}+\frac{y^2}{b^2-\lambda}+\frac{z^2}{c^2-\lambda}=1,$$
and let the equation of the given plane be
$$lx+my+nz=1.$$

The equation of the polar plane of the point (x', y', z') is
$$\frac{xx'}{a^2-\lambda}+\frac{yy'}{b^2-\lambda}+\frac{zz'}{c^2-\lambda}=1.$$

Comparing this equation with the equation of the given plane, we have
$$\frac{x'}{a^2-\lambda}=l,\ \frac{y'}{b^2-\lambda}=m,\ \text{and}\ \frac{z'}{c^2-\lambda}=n;$$
therefore $$\frac{x'}{l}-a^2=\frac{y'}{m}-b^2=\frac{z'}{n}-c^2.$$

Hence the locus of the poles is the straight line whose equations are
$$\frac{x-a^2l}{l}=\frac{y-b^2m}{m}=\frac{z-c^2n}{n}.$$

This straight line is perpendicular to the given plane, and it clearly must pass through the point of contact of that confocal which touches the plane. Hence the perpendicular from any point on its polar plane with respect to a conicoid meets the polar plane in the point where a confocal conicoid touches it.

171. *The axes of the enveloping cone of a conicoid are the normals to the confocals which pass through its vertex.*

Let OP, OQ, OR be the normals at O to the three conicoids which pass through O and are confocal with a given conicoid; and let P, Q, R be on the polar plane of O with respect to the given conicoid.

By the last article, the line OP is the locus of the poles of the plane QOR with respect to the system of confocals. Hence, the pole of the plane QOR with respect to the given conicoid is on the line OP; the pole is also on the plane PQR, because PQR is the polar plane of O and therefore contains the poles of all planes through O. Therefore the point P is the pole of the plane QOR with respect to the given conicoid. Similarly Q and R are the poles of the planes ROP and POQ respectively. *Hence $OPQR$ is a self-polar tetrahedron with respect to the original conicoid.*

Now let any straight line be drawn through P so as to cut the given conicoid in the points A, B and the plane QOR in C. Then [Art. 56] the pencil $O\ \{APBC\}$ is harmonic; and OP and OC are at right angles, hence OP bisects the angle AOB. This shews that OP is an axis of any cone whose vertex is at O, and whose base is a plane section of the conicoid through P. One such cone is the enveloping cone from O to the given conicoid; hence OP is an axis of the enveloping cone. We can shew in a similar manner that OQ and OR are axes of the enveloping cone.

172. *To find in its simplest form the equation of the enveloping cone of a conicoid.*

Let the equation of the conicoid be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

The equation of any tangent plane is

$$lx+my+nz=\sqrt{(a^2l^2+b^2m^2+c^2n^2)}.$$

Hence the direction-cosines of the normal to any tangent plane which passes through the point (x_0, y_0, z_0) satisfy the equation

$$a^2l^2 + b^2m^2 + c^2n^2 - (lx_0 + my_0 + nz_0)^2 = 0.$$

Hence the equation of the reciprocal of the enveloping cone whose vertex is (x_0, y_0, z_0) is

$$a^2x^2 + b^2y^2 + c^2z^2 - (xx_0 + yy_0 + zz_0)^2 = 0 \qquad \text{(i)}.$$

Similarly the equation of the reciprocal of the enveloping cone of the conicoid

$$\frac{x^2}{a^2 - \lambda} + \frac{y^2}{b^2 - \lambda} + \frac{z^2}{c^2 - \lambda} = 1 \qquad \text{(ii)},$$

is $(a^2 - \lambda)\, x^2 + (b^2 - \lambda)\, y^2 + (c^2 - \lambda) z^2 - (xx_0 + yy_0 + zz_0)^2 = 0 \ldots \text{(iii)}.$

It is clear from Art. 60, that the cones (i) and (iii) are co-axial for all values of λ. Hence, since a cone and its reciprocal are co-axial, it follows that all cones which have a common vertex and envelope confocal conicoids are co-axial; and, by considering the three confocals which pass through the vertex, the enveloping cones to which are the tangent planes, we see that the principal planes of the system of cones are the tangent planes to the confocals which pass through their vertex.

The enveloping cones of the three confocals which pass through (x_0, y_0, z_0) are planes, and their reciprocals are straight lines. Hence the three values of λ for which the left side of (iii) is the product of linear factors (which are imaginary) are the three parameters λ_1, λ_2, λ_3 of the confocals through (x_0, y_0, z_0).

But [Art. 77] the three values of λ for which the left side of (iii) is the product of linear factors are the three roots of the discriminating cubic of (i).

Therefore the roots of the discriminating cubic of (i) are λ_1, λ_2, λ_3; so that the equation of the reciprocal of the enveloping cone, when referred to its axes, is

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = 0.$$

Hence the equation of the enveloping cone is

$$\frac{x^2}{\lambda_1} + \frac{y^2}{\lambda_2} + \frac{z^2}{\lambda_3} = 0.$$

Ex. *Find the locus of the vertices of the right circular cones which circumscribe an ellipsoid.*

If a cone be right circular, the reciprocal cone will be right circular. Hence we require the condition that the cone whose equation is

$$a^2x^2+b^2y^2+c^2z^2-(xx_0+yy_0+zz_0)^2=0,$$

may be right circular.

If x_0, y_0, z_0 be all finite, the conditions for a surface of revolution are [Art. 85]
$$a^2-x_0^2+x_0^2=b^2-y_0^2+y_0^2=c^2-z_0^2+z_0^2,$$
so that, unless the surface is a sphere, $x_0y_0z_0$ must be zero. If $z_0=0$, the condition for a surface of revolution gives

$$(c^2-a^2+x_0^2)(c^2-b^2+y_0^2)=x_0^2y_0^2.$$

Hence the enveloping cone from any point on the focal ellipse

$$\frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2}=1,\ z=0 \text{.............................(i)},$$

is right circular.

Similarly, the enveloping cones from points on

$$\frac{x^2}{a^2-b^2}+\frac{z^2}{c^2-b^2}=1,\ y=0 \text{.............................(ii)},$$

or from points on
$$\frac{y^2}{b^2-a^2}+\frac{z^2}{c^2-a^2}=1,\ x=0 \text{.............................(iii)},$$

are right circular.

The conic (ii) is the focal hyperbola, and (iii) is imaginary.

Concyclic Conicoids.

173. The reciprocal of the conicoid

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1,$$

with respect to the sphere $x^2+y^2+z^2=\kappa^2$, is

$$(a^2+\lambda)\,x^2+(b^2+\lambda)\,y^2+(c^2+\lambda)\,z^2=\kappa^4.$$

It is clear that the reciprocal conicoids have the same cyclic planes for all values of λ.

Hence a system of confocal conicoids reciprocates into a system of concyclic conicoids.

174. The following are examples of reciprocal properties of confocal and concyclic conicoids.

Three confocals pass through any point, namely an ellipsoid, an hyperboloid of one sheet, and an hyperboloid of two sheets; also the tangent planes at the point to the three surfaces are at right angles.

Three concyclics touch any plane, namely an ellipsoid, an hyperboloid of one sheet, and an hyperboloid of two sheets; also the lines from the centre to the points of contact of the plane are at right angles.

Two confocals touch a straight line, and the tangent planes at the points of contact are at right angles.

Two concyclics touch a straight line, and the lines from the centre to the points of contact are at right angles.

One conicoid of a confocal system touches any plane.

One conicoid of a concyclic system passes through any point.

The locus of the pole of a given plane with respect to a system of confocals is a straight line.

The envelope of the polar plane of a given point with respect to a system of concyclics is a straight line.

The principal planes of a cone enveloping a conicoid are the tangent planes to the confocals through its vertex.

The axes of a cone whose vertex is at the centre of a conicoid and base any plane section, are the lines from the centre to the points of contact of the plane with the concyclics which touch it.

Foci of Conicoids.

175. There are two definitions of a conicoid which correspond to the focus and directrix definition of a conic.

One definition, due to Mac Cullagh, is as follows :—

A conicoid is the locus of a point which moves so that its distance from a fixed point, called the focus, is in a constant ratio to its distance (measured parallel to a fixed plane) from a fixed straight line called the directrix.

Let the origin be the focus, and the plane $z=0$ the fixed plane.

Also let the equations of the directrix be

$$\frac{x-f}{l}=\frac{y-g}{m}=\frac{z-h}{n}.$$

Let x', y', z' be the co-ordinates of any point P on the locus, and let a plane through P parallel to $z=0$ meet the directrix in M, then M is $\left\{f+\frac{l}{n}(z'-h),\ g+\frac{m}{n}(z'-h),\ z'\right\}$.

Now $OP^2=e^2.PM^2$, e being the constant ratio. Hence the equation of the locus of (x', y', z') is

$$x^2+y^2+z^2=e^2\left[\left\{x-f-\frac{l}{n}(z-h)\right\}^2+\left\{y-g-\frac{m}{n}(z-h)\right\}^2\right]\ldots\text{(i)}.$$

The locus is therefore a conicoid, and is such that sections parallel to $z=0$ are circles.

If the axes be changed in any manner (i) will always be of the form

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2-A=0,$$

where A is the sum of two squares, or is the product of two imaginary factors. We can therefore find the foci of any given conicoid whose equation is $S=0$, from the consideration that $S-\lambda\{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2\}$ will be the product of imaginary linear factors if (α, β, γ) be a focus, provided a suitable value be given to λ.

176. The other definition of a conicoid, due to Salmon, is as follows :—

A conicoid is the locus of a point the square of whose distance from a fixed point, called a focus, varies as the product of its distances from two fixed planes.

The equation of the locus is clearly of the form

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=k^2(lx+my+nz+p)(l'x+m'y+n'z+p').$$

We can find the foci of any conicoid according to this definition by the consideration that

$$S-\lambda\{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2\}$$

will be the product of real linear factors if (α, β, γ) be a focus, provided a suitable value be given to λ.

177. *To find the foci of the conicoid whose equation is*

$$ax^2 + by^2 + cz^2 = 1.$$

We have seen in Articles 175 and 176 that (α, β, γ) is a focus when

$$ax^2 + by^2 + cz^2 - 1 - \lambda\{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2\} \ldots\ldots \text{(i)}$$

is the product of linear factors.

Hence λ must be equal to a, or b, or c.

Let $\lambda = a$, then (i) becomes

$$(b-a)\,y^2 + (c-a)\,z^2 + 2a\alpha x + 2a\beta y + 2a\gamma z - a\,(\alpha^2 + \beta^2 + \gamma^2) - 1,$$

or $$(b-a)\left\{y + \frac{a\beta}{b-a}\right\}^2 + (c-a)\left\{z + \frac{a\gamma}{c-a}\right\}^2$$

$$+ 2a\alpha x - a\alpha^2 - \frac{ab\beta^2}{b-a} - \frac{ac\gamma^2}{c-a} - 1.$$

Hence, in order that (i) may be the product of linear factors, we must have $\alpha = 0$, and

$$\frac{\beta^2}{\frac{1}{b} - \frac{1}{a}} + \frac{\gamma^2}{\frac{1}{c} - \frac{1}{a}} = 1.$$

Similarly, if $= b$, we have $\beta = 0$ and

$$\frac{\alpha^2}{\frac{1}{a} - \frac{1}{b}} + \frac{\gamma^2}{\frac{1}{c} - \frac{1}{b}} = 1;$$

and, if $\lambda = c$, we have $\gamma = 0$, and

$$\frac{\alpha^2}{\frac{1}{a} - \frac{1}{c}} + \frac{\beta^2}{\frac{1}{b} - \frac{1}{c}} = 1.$$

There are therefore three conics, one in each principal plane, on which the foci lie.

178. If the surface be an ellipsoid whose semiaxes are a, b, c, the conics on which the foci lie are

$$\frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2}=1,\ z=0\ldots\ldots\ldots\ldots\ldots\text{(i)},$$

$$\frac{x^2}{a^2-b^2}+\frac{z^2}{c^2-b^2}=1,\ y=0\ldots\ldots\ldots\ldots\ldots\text{(ii)},$$

and
$$\frac{y^2}{b^2-a^2}+\frac{z^2}{c^2-a^2}=1,\ x=0\ldots\ldots\ldots\ldots\ldots\text{(iii)}.$$

Since a, b, c are in descending order of magnitude (i) is an ellipse, (ii) is an hyperbola, and (iii) is imaginary. These conics are called the *focal conics;* and, as we have seen in Art. 160, they are the boundaries of limiting forms of confocal conicoids.

179. The focal conics of the cone $ax^2+by^2+cz^2=0$ can be deduced from the above, or found in a similar manner. The conics become

$$x=0,\ \frac{y^2}{\frac{1}{b}-\frac{1}{a}}+\frac{z^2}{\frac{1}{c}-\frac{1}{a}}=0;$$

$$y=0,\ \frac{z^2}{\frac{1}{c}-\frac{1}{b}}+\frac{x^2}{\frac{1}{a}-\frac{1}{b}}=0;$$

and
$$z=0,\ \frac{x^2}{\frac{1}{a}-\frac{1}{c}}+\frac{y^2}{\frac{1}{b}-\frac{1}{c}}=0.$$

One of the focal conics of a cone is therefore a pair of real straight lines which are called the *focal lines;* the other focal conics are pairs of imaginary straight lines, which we may consider as point-ellipses.

Ex. 1. Two cones which have the same focal lines cut one another at right angles.

Ex. 2. Shew that the enveloping cones from any point to a system of confocals have the same focal lines.

Ex. 3. Shew that the focal conics of a paraboloid are two parabolas.

180. *The focal lines of a cone are perpendicular to the cyclic planes of the reciprocal cone.*

The equations of any two reciprocal cones referred to their axes are

$$ax^2 + by^2 + cz^2 = 0, \text{ and } \frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0.$$

The cyclic planes are [Art. 121]

$$(a - b)\,x^2 + (c - b)\,z^2 = 0, \text{ and } \left(\frac{1}{a} - \frac{1}{b}\right)x^2 + \left(\frac{1}{c} - \frac{1}{b}\right)z^2 = 0.$$

The focal lines are by the last article

$$y = 0,\ \frac{x^2}{\frac{1}{a} - \frac{1}{b}} + \frac{z^2}{\frac{1}{c} - \frac{1}{b}} = 0, \text{ and } y = 0,\ \frac{x^2}{a - b} + \frac{z^2}{c - b} = 0.$$

It is therefore clear that the focal lines of one cone are perpendicular to the cyclic planes of the other.

Examples on Chapter VIII.

1. Three confocal conicoids meet in a point, and a central plane of each is drawn parallel to its tangent plane at that point. Prove that, one of the three sections will be an ellipse, one an hyperbola, and one imaginary.

2. Plane sections of an ellipsoid envelope a confocal; shew that their centres lie on a surface of the fourth degree.

3. P, Q are two points on a generator of a hyperboloid; P', Q' the corresponding points on a confocal hyperboloid. Shew that $P'Q'$ is a generator of the latter, and that $PQ = P'Q'$.

4. Shew that the points on a system of confocals which are such that the normals are parallel to a given line are on a rectangular hyperbola.

5. If three lines at right angles to one another touch a conicoid, the plane through the points of contact will envelope a confocal.

6. If three of the generating lines of the enveloping cone of a paraboloid be mutually at right angles, shew that the vertex will be on a paraboloid, and that the polar plane of the vertex will always touch another paraboloid.

7. If through a given straight line tangent planes be drawn to a system of confocals, the corresponding normals generate a hyperbolic paraboloid.

8. Shew that the locus of the polar of a given line with respect to a system of confocals is a hyperbolic paraboloid one of whose asymptotic planes is perpendicular to the given line.

9. Planes are drawn all passing through a fixed straight line and each touching one of a set of confocal ellipsoids; find the locus of their points of contact.

10. At a given point O the tangent planes to the three conicoids which pass through O, and are confocal with a given conicoid, are drawn; shew that these tangent planes and the polar plane of O form a tetrahedron which is self-conjugate with respect to the given conicoid.

11. Through a straight line in one of the principal planes tangent planes are drawn to a series of confocal ellipsoids; prove that the points of contact lie on a plane, and that the normals at these points pass through a fixed point.

If a plane be drawn cutting the three principal planes, and through each of the lines of section tangent planes be drawn to the series of conicoids, prove that the three planes which are the loci of the points of contact intersect in a straight line which is perpendicular to the cutting plane, and passes through the three fixed points in which the three series of normals intersect.

12. Any tangent plane to a cone makes equal angles with the planes through the line of contact and the focal lines.

13. If through a tangent at any point of a conicoid two tangent planes be drawn to a focal conic, these two planes will be equally inclined to the tangent plane at O.

14. The focal lines of the enveloping cone of a conicoid are the generating lines of the confocal hyperboloid of one sheet which passes through its vertex.

15. Any section of a cone which is normal at P to a focal line, has P for one focus.

16. If a section of an ellipsoid be normal to a focal conic at P, then P will be a focus of the section.

17. The product of the distances of any point P on a focal conic of an ellipsoid, from two tangent planes to the surface which are parallel to one another and to the tangent at P to the focal conic, is constant for all positions of P.

18. From whatever point in space the two focal conics are viewed they appear to cut at right angles.

Hence shew that the focal conics project into confocals on any plane.

19. If two confocal surfaces be viewed from any point, their apparent contours seem to cut at right angles.

20. If two cylinders with parallel generators circumscribe confocal surfaces their sections by a plane perpendicular to the generators are confocal conics.

21. The centres of the sections of a series of confocal conicoids by a given plane lie on a straight line.

22. Shew that those tangent lines to an ellipsoid from an external point whose length is a maximum or minimum are normals at their respective points of contact to confocals drawn through those points: and further, that the locus of these maximum and minimum lines to a series of ellipsoids confocal with the original one is a cone of the second degree.

23. A straight line meets a quadric in two points P, Q so that the normals at P and Q intersect: prove that PQ meets any confocal quadric in points, the normals at which intersect, and that if PQ pass through a fixed point it lies on a quadric cone.

24. If from any point O normals are drawn to a system of confocals (1) these normals form a cone of the second degree, (2) the tangent planes at the feet of the normals form a developable of the fourth degree. Consider the case of O being in one of the principal planes.

25. The envelope of the polar plane of a fixed point with respect to a system of confocal quadrics is a developable surface. Prove this, and shew that the developable surface touches the six tangent planes to any one of the confocals at the points where the normals to that confocal through the fixed point meet that confocal.

26. Prove that the developable which is the envelope of the polar planes of a fixed point P with respect to a system of confocal quadrics, meet Q the polar plane of P with respect to one of the confocals in a line, whose polar line with respect to the same confocal is perpendicular to Q; and that these polar lines generate the quadric cone six of whose generators are the normals at P to the three confocals through P, and the three lines through P parallel to their axes.

27. Prove that if a model of a hyperboloid of one sheet be constructed of rods representing the generating lines, jointed at the points of crossing; then if the model be deformed it will assume the form of a confocal hyperboloid, and prove that the trajectory of a point on the model will be orthogonal to the system of confocal hyperboloids.

28. The two quadrics

$$2ayz + 2bzx + 2cxy = 1 \text{ and } 2a'yz + 2b'zx + 2c'xy = 1$$

can be placed so as to be confocal if

$$\frac{abc}{a^2+b^2+c^2} + \frac{a'b'c'}{a'^2+b'^2+c'^2} = 0, \text{ and } \frac{a^2b^2c^2}{(a^2+b^2+c^2)^3} + \frac{a'^2b'^2c'^2}{(a'^2+b'^2+c'^2)^3} = \tfrac{1}{27}.$$

29. Two ellipsoids, two hyperboloids of one sheet, and two hyperboloids of two sheets belong to the same confocal system; shew that of the 256 straight lines joining a point of intersection of three surfaces to a point of intersection of the other three, there are 8 sets of 32 equal lines, the lines of each set agreeing either in crossing or in not crossing each of the principal planes.

30. A variable conicoid has double contact with each of three fixed confocals; shew that it has a fixed director-sphere.

CHAPTER IX.

QUADRIPLANAR AND TETRAHEDRAL CO-ORDINATES.

181. In the quadriplanar system of co-ordinates, four planes, which form a tetrahedron, are taken as planes of reference, and the co-ordinates of any point are its perpendicular distances from the four planes. The perpendiculars are considered positive when they are drawn in the same direction as the perpendiculars from the opposite angular points of the tetrahedron.

Since the perpendicular distances of a point from any *three* planes are sufficient to determine its position, there must be some relation connecting the four perpendiculars on the planes of reference.

Let A, B, C, D be the angular points of the tetrahedron, and a, b, c, d be the areas of the faces opposite respectively to A, B, C, D; then, if $\alpha, \beta, \gamma, \delta$ be the co-ordinates of any point, the relation will be

$$a\alpha + b\beta + c\gamma + d\delta = 3V,$$

where V is the volume of the tetrahedron $ABCD$. This is evidently true for any point P within the tetrahedron, since the sum of the tetrahedra $BCDP$, $CDAP$, $DABP$, $ABCP$ is the tetrahedron $ABCD$; and, regard being had to the signs of the perpendiculars, it can be easily seen to be universally true.

182. The tetrahedral co-ordinates $\alpha, \beta, \gamma, \delta$ of any point P are the ratios of the tetrahedra $BCDP$, $CDAP$, $DABP$, $ABCP$ to the tetrahedron of reference $ABCD$. The relation between the co-ordinates is easily seen to be

$$\alpha+\beta+\gamma+\delta=1.$$

It is generally immaterial whether we use quadriplanar or tetrahedral co-ordinates, but the latter system has some advantages, and in what follows we shall always suppose the co-ordinates to be tetrahedral unless the contrary is stated.

We shall also suppose that the equations are homogeneous, for they can clearly always be made so by means of the relation $\alpha+\beta+\gamma+\delta=1$. When the equations are homogeneous we can use instead of the actual co-ordinates any quantities proportional to them.

183. The co-ordinates of the point which divides the line joining $(\alpha_1, \beta_1, \gamma_1, \delta_1)$ and $(\alpha_2, \beta_2, \gamma_2, \delta_2)$ in the ratio $\lambda : \mu$ are easily seen to be

$$\frac{\mu\alpha_1+\lambda\alpha_2}{\lambda+\mu}, \quad \frac{\mu\beta_1+\lambda\beta_2}{\lambda+\mu}, \quad \frac{\mu\gamma_1+\lambda\gamma_2}{\lambda+\mu}, \quad \frac{\mu\delta_1+\lambda\delta_2}{\lambda+\mu}.$$

184. *The general equation of the first degree represents a plane.*

The general equation of the first degree is

$$l\alpha+m\beta+n\gamma+p\delta=0.$$

We may shew that this represents a plane by the method of Art. 13.

Since the equation $l\alpha+m\beta+n\gamma+p\delta=0$ contains three independent constants it is the most general form of the equation of a plane.

The equation of the plane through the three points $(\alpha_1, \beta_1, \gamma_1, \delta_1)$, $(\alpha_2, \beta_2, \gamma_2, \delta_2)$, $(\alpha_3, \beta_3, \gamma_3, \delta_3)$ is

$$\begin{vmatrix} \alpha, & \beta, & \gamma, & \delta \\ \alpha_1, & \beta_1, & \gamma_1, & \delta_1 \\ \alpha_2, & \beta_2, & \gamma_2, & \delta_2 \\ \alpha_3, & \beta_3, & \gamma_3, & \delta_3 \end{vmatrix} = 0.$$

185. *To shew that the perpendiculars from the angular points of the tetrahedron of reference on the plane whose equation is* $l\alpha + m\beta + n\gamma + p\delta = 0$ *are proportional to* l, m, n, p.

Let L, M, N, P be the perpendiculars on the plane from the angular points A, B, C, D respectively; the perpendiculars being estimated in the same direction. Let the plane meet the edge AB in K; then at K we have $\gamma = 0$, $\delta = 0$ and $l\alpha + m\beta = 0$; therefore $\dfrac{\alpha}{m} = -\dfrac{\beta}{l}$.

Now $$L : M :: AK : BK.$$

But $$AK : AB :: ACDK : ACDB :: \beta : 1;$$

similarly $$KB : AB :: KBCD : ABCD :: \alpha : 1;$$

$$\therefore\ L : M :: AK : -KB :: \beta : -\alpha :: l : m;$$

$$\therefore\ \frac{L}{l} = \frac{M}{m}, \text{ and similarly each} = \frac{N}{n} = \frac{P}{p}.$$

186. The lengths of the perpendiculars on a plane from the vertices of the tetrahedron of reference may be called the *tangential co-ordinates* of the plane; and, from the preceding article, the equation of the plane whose tangential co-ordinates are l, m, n, p is $l\alpha + m\beta + n\gamma + p\delta = 0$.

The co-ordinates of all planes which pass through the point whose tetrahedral co-ordinates are α_1, β_1, γ_1, δ_1, are connected by the relation $l\alpha_1 + m\beta_1 + n\gamma_1 + p\delta_1 = 0$. Hence the tangential equation of a point is of the first degree.

187. The equation of any plane through the intersection of the two planes whose equations are

$$l\alpha + m\beta + n\gamma + p\delta = 0, \text{ and } l'\alpha + m'\beta + n'\gamma + p'\delta = 0,$$

is $$(l + \lambda l')\,\alpha + (m + \lambda m')\,\beta + (n + \lambda n')\,\gamma + (p + \lambda p')\,\delta = 0.$$

Hence the tangential co-ordinates of any plane through the line of intersection of the two planes whose co-ordinates are l, m, n, p and l', m', n', p' are *proportional* to $l + \lambda l'$, $m + \lambda m'$, $n + \lambda n'$, $p + \lambda p'$.

188. *To find the perpendicular distance of a point from a plane.*

Let the equation of the plane be

$$l\alpha + m\beta + n\gamma + p\delta = 0 \qquad \text{(i)},$$

and let its equation referred to any three perpendicular axes be

$$Ax + By + Cz + D = 0 \qquad \text{(ii)}.$$

We know that the perpendicular distance of any point from the plane (ii) is proportional to the result obtained by substituting the co-ordinates of the point in the left-hand member of the equation. Hence the perpendicular distance of any point from (i) is proportional to the result obtained by substituting the co-ordinates in the expression

$$l\alpha + m\beta + n\gamma + p\delta.$$

Hence, if l, m, n, p be *equal* to the lengths of the perpendiculars from the angular points of the tetrahedron of reference, the perpendicular distance of any other point $(\alpha', \beta', \gamma', \delta')$ will be $l\alpha' + m\beta' + n\gamma' + p\delta'$.

189. If a plane be at an infinite distance from the angular points of the tetrahedron of reference, the perpendiculars upon it from those points are all equal.

Hence the equation of the plane at infinity is

$$\alpha + \beta + \gamma + \delta = 0.$$

This result may also be obtained in the following manner.

Let $k\alpha, k\beta, k\gamma, k\delta$ be the co-ordinates of any point; then the invariable relation gives $k\alpha + k\beta + k\gamma + k\delta = 1$, or $\alpha + \beta + \gamma + \delta = \frac{1}{k}$. If therefore k become infinitely great, we have in the limit $\alpha + \beta + \gamma + \delta = 0$. This is the relation which is satisfied by finite quantities that are proportional to the co-ordinates of *any* infinitely distant point.

190. Let $\alpha_1, \beta_1, \gamma_1, \delta_1$ be the co-ordinates of any point P, and $\alpha, \beta, \gamma, \delta$ the co-ordinates of a point Q. Also let $\theta_1, \theta_2, \theta_3, \theta_4$

be respectively the angles between the line PQ and the perpendiculars from the angular points A, B, C, D of the fundamental tetrahedron on the opposite faces.

Then, a, b, c, d being the areas of the faces opposite to A, B, C, D respectively, we have

$$\alpha-\alpha_1=\tfrac{1}{3}a.PQ\cos\theta_1,\quad \beta-\beta_1=\tfrac{1}{3}b.PQ\cos\theta_2,$$
$$\gamma-\gamma_1=\tfrac{1}{3}c.PQ\cos\theta_3,\text{ and }\delta-\delta_1=\tfrac{1}{3}d\,.PQ\cos\theta_4.$$

The equations of the straight line through P, whose direction-angles are θ_1, θ_2, θ_3, θ_4, are therefore

$$\frac{\alpha-\alpha_1}{a\cos\theta_1}=\frac{\beta-\beta_1}{b\cos\theta_2}=\frac{\gamma-\gamma_1}{c\cos\theta_3}=\frac{\delta-\delta_1}{d\cos\theta_4}=\tfrac{1}{3}r.$$

Since the sum of the projections of the four faces of the tetrahedron on a plane perpendicular to PQ is zero, we have

$$a\cos\theta_1+b\cos\theta_2+c\cos\theta_3+d\cos\theta_4=0,$$

or, putting l, m, n, p instead of $a\cos\theta_1$, $b\cos\theta_2$, $c\cos\theta_3$, $d\cos\theta_4$ respectively,

$$l+m+n+p=0.$$

Ex. 1. Find the conditions that three planes may have a common line of intersection.

Ex. 2. Find the conditions that two planes may be parallel.

Ex. 3. Find the equation of a plane through a given point parallel to a given plane.

[Any plane parallel to $l\alpha+m\beta+n\gamma+p\delta=0$, is

$$l\alpha+m\beta+n\gamma+p\delta+\lambda(\alpha+\beta+\gamma+\delta)=0.$$

Hence the parallel plane through $(\alpha', \beta', \gamma', \delta')$ is

$$l\alpha+m\beta+n\gamma+p\delta=(l\alpha'+m\beta'+n\gamma'+p\delta')(\alpha+\beta+\gamma+\delta).]$$

Ex. 4. The equations of the four planes each of which passes through a vertex of the tetrahedron of reference and is parallel to the opposite face are

$$\beta+\gamma+\delta=0,\ \gamma+\delta+\alpha=0,\ \delta+\alpha+\beta=0,\text{ and }\alpha+\beta+\gamma=0.$$

Ex. 5. Find the condition that four given points may lie on a plane.

Ex. 6. Find the condition that four given planes may meet in a point.

Ex. 7. The equations of the four planes each of which bisects three of the edges of a tetrahedron are

$$\alpha=\beta+\gamma+\delta,\ \beta=\gamma+\delta+\alpha,\ \gamma=\delta+\alpha+\beta,\text{ and }\delta=\alpha+\beta+\gamma.$$

Ex. 8. Shew that the lines joining the middle points of opposite edges of a tetrahedron meet in a point.

Ex. 9. Find the equations of the four lines through A, B, C, D respectively parallel to the line whose equations are

$$l\alpha+m\beta+n\gamma+p\delta=0, \quad l'\alpha+m'\beta+n'\gamma+p'\delta=0.$$

Ex. 10. A plane cuts the edges of a tetrahedron in six points, and six other points are taken, one on each edge, so that each edge is divided harmonically: shew that the six planes each of which passes through one of the six latter points and through the edge opposite to it, will meet in a point.

Ex. 11. Lines AOa, BOb, COc, DOd through the angular points of a tetrahedron meet the opposite faces in a, b, c, d. Shew that the four lines of intersection of the planes BCD, bcd; CDA, cda; DAB, dab; and ABC, abc lie on a plane.

[If O be $(\alpha', \beta', \gamma', \delta')$ the equation of bcd is

$$\frac{\beta}{\beta'}+\frac{\gamma}{\gamma'}+\frac{\delta}{\delta'}-\frac{2\alpha}{\alpha'}=0;$$

hence the line of intersection of BCD, bcd is on the plane

$$\frac{\alpha}{\alpha'}+\frac{\beta}{\beta'}+\frac{\gamma}{\gamma'}+\frac{\delta}{\delta'}=0.]$$

Ex. 12. If two tetrahedra be such that the straight lines joining corresponding angular points meet in a point, then will the four lines of intersection of corresponding faces lie on a plane.

191. We shall write the general equation of the second degree in tetrahedral co-ordinates in the form

$$q\alpha^2+r\beta^2+s\gamma^2+t\delta^2+2f\beta\gamma+2g\gamma\alpha+2h\alpha\beta$$
$$+2u\alpha\delta+2v\beta\delta+2w\gamma\delta=0.$$

The left side of the equation will be denoted by $F(\alpha, \beta, \gamma, \delta)$.

192. *To find the points where a given straight line cuts the surface represented by the general equation of the second degree in tetrahedral co-ordinates.*

Let the equations of the straight line be

$$\frac{\alpha-\alpha_1}{l}=\frac{\beta-\beta_1}{m}=\frac{\gamma-\gamma_1}{n}=\frac{\delta-\delta_1}{p}=\rho.$$

To find the points common to this line and the surface, we have the equation

$$F(\alpha_1 + l\rho,\ \beta_1 + m\rho,\ \gamma_1 + n\rho,\ \delta_1 + p\rho) = 0,$$

or $$F(\alpha_1, \beta_1, \gamma_1, \delta_1) + \rho\left(l\frac{dF}{d\alpha_1} + m\frac{dF}{d\beta_1} + n\frac{dF}{d\gamma_1} + p\frac{dF}{d\delta_1}\right) + \rho^2 F(l, m, n, p) = 0.$$

Since there are two values of ρ, the surface is a conicoid.

193. *To find the equation of a tangent plane at any point of a conicoid.*

If $(\alpha_1, \beta_1, \gamma_1, \delta_1)$ be a point on the surface, one root of the equation found in the preceding article will be zero. Two roots will be zero, if

$$l\frac{dF}{d\alpha_1} + m\frac{dF}{d\beta_1} + n\frac{dF}{d\gamma_1} + p\frac{dF}{d\delta_1} = 0.$$

The line will in that case be a tangent line to the surface.

Substituting for l, m, n, p from the equations of the straight line, we obtain the equation of the tangent plane, namely

$$(\alpha - \alpha_1)\frac{dF}{d\alpha_1} + (\beta - \beta_1)\frac{dF}{d\beta_1} + (\gamma - \gamma_1)\frac{dF}{d\gamma_1} + (\delta - \delta_1)\frac{dF}{d\delta_1} = 0.$$

But, since the equation $F(\alpha, \beta, \gamma, \delta) = 0$ is homogeneous,

$$\alpha_1\frac{dF}{d\alpha_1} + \beta_1\frac{dF}{d\beta_1} + \gamma_1\frac{dF}{d\gamma_1} + \delta_1\frac{dF}{d\delta_1} = 0;$$

therefore the equation of the tangent plane at the point $(\alpha_1, \beta_1, \gamma_1, \delta_1)$ is

$$\alpha\frac{dF}{d\alpha_1} + \beta\frac{dF}{d\beta_1} + \gamma\frac{dF}{d\gamma_1} + \delta\frac{dF}{d\delta_1} = 0.$$

194. It can be shewn by the method of Art. 53, that the equation of the polar plane of any point $(\alpha_1, \beta_1, \gamma_1, \delta_1)$ is

$$\alpha\frac{dF}{d\alpha_1} + \beta\frac{dF}{d\beta_1} + \gamma\frac{dF}{d\gamma_1} + \delta\frac{dF}{d\delta_1} = 0.$$

195. *To find the co-ordinates of the centre of the conicoid.*

The polar plane of the centre is the plane at infinity, whose equation is $\alpha + \beta + \gamma + \delta = 0$.

Hence, if $(\alpha_1, \beta_1, \gamma_1, \delta_1)$ be the centre of the conicoid, we must have

$$\frac{dF}{d\alpha_1} = \frac{dF}{d\beta_1} = \frac{dF}{d\gamma_1} = \frac{dF}{d\delta_1}.$$

196. The diametral plane of a system of parallel chords of the conicoid can be found from Art. 192. The equation of the plane is

$$l\frac{dF}{d\alpha} + m\frac{dF}{d\beta} + n\frac{dF}{d\gamma} + p\frac{dF}{d\delta} = 0.$$

Since $l + m + n + p = 0$ [Art. 190], it follows that all the diametral planes pass through the centre, that is through the point for which

$$\frac{dF}{d\alpha} = \frac{dF}{d\beta} = \frac{dF}{d\gamma} = \frac{dF}{d\delta}.$$

197. *To find the condition that a given plane may touch the conicoid.*

The condition that the plane $l\alpha + m\beta + n\gamma + p\delta = 0$ may touch the conicoid can be found as in Art. 57. The result is

$$Ql^2 + Rm^2 + Sn^2 + Tp^2 + 2Fmn + 2Gnl$$
$$+ 2Hlm + 2Ulp + 2Vmp + 2Wnp = 0,$$

where Q, R, S &c. are the co-factors of q, r, s &c. in the discriminant.

198. *To find the condition that the surface represented by the general equation of the second degree may be a cone.*

The polar planes of the angular points of the fundamental tetrahedron with respect to a cone meet in a point, namely in the vertex of the cone. The equations of the polar planes are

$$q\alpha + h\beta + g\gamma + u\delta = 0,$$
$$h\alpha + r\beta + f\gamma + v\delta = 0,$$
$$g\alpha + f\beta + s\gamma + w\delta = 0,$$

and $$u\alpha + v\beta + w\gamma + t\delta = 0.$$

The required condition is therefore

$$\begin{vmatrix} q, & h, & g, & u \\ h, & r, & f, & v \\ g, & f, & s, & w \\ u, & v, & w, & t \end{vmatrix} = 0.$$

199. *To shew that any two conicoids have a common self-polar tetrahedron.*

We can shew, as in Art. 142, that four cones can pass through the intersection of any two conicoids, and that the vertices of the four cones are the angular points of a tetrahedron self-polar with respect to any conicoid through the curve of intersection of the given conicoids.

The equation of a conicoid, when referred to a self-polar tetrahedron, takes the form

$$q\alpha^2 + r\beta^2 + s\gamma^2 + t\delta^2 = 0.$$

For, since $\alpha = 0$ is the polar plane of the point (1, 0, 0, 0), we have $h = g = u = 0$; and similarly $f = v = w = 0$.

200. *To find the general equation of a conicoid circumscribing the tetrahedron of reference.*

If we substitute the co-ordinates of the angular points of the tetrahedron of reference in the general equation of the second degree, we have the conditions $q = r = s = t = 0$.

Hence the general equation of a conicoid circumscribing the tetrahedron of reference is

$$f\beta\gamma + g\gamma\alpha + h\alpha\beta + u\alpha\delta + v\beta\delta + w\gamma\delta = 0.$$

201. *To find the general equation of a conicoid which touches the faces of the tetrahedron of reference.*

The planes $\alpha = 0$, $\beta = 0$, $\gamma = 0$ and $\delta = 0$ will touch the conicoid given by the general equation of the second degree if $Q = 0$, $R = 0$, $S = 0$ and $T = 0$. [Art. 197.]

Hence conicoids which are inscribed in the tetrahedron of reference are given by the general equation, with the conditions $Q = R = S = T = 0$.

Ex. 1. Find the equation of a conicoid which circumscribes the tetrahedron of reference, and is such that the tangent planes at the angular points are parallel to the opposite faces. *Ans.* $\beta\gamma+\gamma\alpha+\alpha\beta+\alpha\delta+\beta\delta+\gamma\delta=0.$

Ex. 2. Find the equation of the conicoid which touches each of the faces of the fundamental tetrahedron at its centre of gravity.

Ans. $\alpha^2+\beta^2+\gamma^2+\delta^2-\beta\gamma-\gamma\alpha-\alpha\beta-\alpha\delta-\beta\delta-\gamma\delta=0.$

202. *To find the equation of the sphere which circumscribes the tetrahedron of reference.*

The general equation of a circumscribing conicoid is

$$f\beta\gamma+g\gamma\alpha+h\alpha\beta+u\alpha\delta+v\beta\delta+w\gamma\delta=0.$$

If the conicoid be the circumscribing sphere, the section by $\delta=0$ will be the circle circumscribing the triangle ABC. Now the triangular co-ordinates of any point in the plane $\delta=0$, referred to the triangle ABC, are clearly the same as the tetrahedral co-ordinates of that point, referred to the tetrahedron $ABCD$. Hence, when we put $\delta=0$ in the equation of the conicoid, we shall obtain an equation of the same form as the triangular equation of the circle circumscribing ABC. Hence, comparing the equations

$$f\beta\gamma+g\gamma\alpha+h\alpha\beta=0,$$

and
$$BC^2\beta\gamma+CA^2\gamma\alpha+AB^2\alpha\beta=0,$$

we obtain
$$\frac{f}{BC^2}=\frac{g}{CA^2}=\frac{h}{AB^2}.$$

By considering the sections made by the other faces of the tetrahedron, we obtain the equation of the circumscribing sphere in the form

$$BC^2\beta\gamma+CA^2\gamma\alpha+AB^2\alpha\beta+AD^2\alpha\delta+BD^2\beta\delta+CD^2\gamma\delta=0.$$

203. *To find the conditions that the general equation of the second degree may represent a sphere.*

Since the terms of the second degree in the equations of all spheres, referred to rectangular axes, are the same; if $S=0$ be the equation of any one sphere, the equation of any other sphere can be written in the form

$$S+l\alpha+m\beta+n\gamma+p\delta=0,$$

or, in the homogeneous form,

$$S+(l\alpha+m\beta+n\gamma+p\delta)(\alpha+\beta+\gamma+\delta)=0.$$

If this be the same conicoid as that given by the general equation of the second degree, $S=0$ being the equation of the circumscribing sphere found in Art. 202, we must have, for some value of λ,

$$\lambda q = l, \quad \lambda r = m, \quad \lambda s = n, \quad \lambda t = p;$$

also
$$2\lambda f = BC^2 + m + n,$$

and five similar equations.

Hence the required conditions are that $\dfrac{r+s-2f}{BC^2}$ should be equal to the five similar expressions.

The conditions for a sphere may also be obtained by means of the equation found in Art. 192; or in the following manner.

To find the points, P_1, P_2 suppose, where the edge BC meets the conicoid given by the general equation of the second degree, we must put $\alpha = 0$, $\delta = 0$; and we obtain

$$r\beta^2 + s\gamma^2 + 2f\beta\gamma = 0;$$

we have also
$$\beta + \gamma = 1;$$

$$\therefore \; r\beta^2 + s(1-\beta)^2 + 2f\beta(1-\beta) = 0,$$

and, if the roots be β_1, β_2, we have

$$\beta_1\beta_2 = \frac{s}{r+s-2f}.$$

Now
$$\beta_1\beta_2 = \frac{CP_1 \,.\, CP_2}{BC^2};$$

hence, if the conicoid be a sphere, and if t_1, t_2, t_3, t_4 be the lengths of the tangents from the points A, B, C, D respectively, we have

$$\frac{r+s-2f}{BC^2} = \frac{s}{t_3^2}.$$

By considering the edges CD, CA we have similarly

$$\frac{s+t-2w}{CD^2} = \frac{q+s-2g}{CA^2} = \frac{s}{t_3^2}.$$

Hence, as above, the required conditions are that $\dfrac{r+s-2f}{BC^2}$ should be equal to the similar expressions.

Examples on Chapter IX.

1. Shew that, if $q\alpha^2 + r\beta^2 + s\gamma^2 + t\delta^2 = 0$ be a paraboloid, it will touch the eight planes $\alpha \pm \beta \pm \gamma \pm \delta = 0$.

2. The locus of the pole of a given plane with respect to a system of conicoids which touch eight fixed planes is a straight line.

3. The polar planes of a given point, with respect to a system of conicoids which pass through eight given points, all pass through a straight line.

4. If two pairs of the opposite edges of a tetrahedron are each to each at right angles to one another, the remaining pair will be at right angles. Shew also that in this case the middle points of the six edges lie on a sphere.

5. Shew that an ellipsoid may be described so as to touch each edge of any tetrahedron in its middle point.

6. If six points are taken one on each edge of a tetrahedron such that the three lines joining the points on opposite edges meet in a point, then will a conicoid touch the edges at those points.

7. If two conicoids touch the edges of a tetrahedron, the twelve points of contact are on another conicoid.

8. If a conicoid touch the edges of a tetrahedron, the lines joining the angular points of the tetrahedron and of the polar tetrahedron will meet in a point.

9. Shew that any two conicoids, and the polar reciprocal of each with respect to the other have a common self-polar tetrahedron.

10. A series of conicoids U_1, U_2, U_3... are such that U_{r+1} and U_{r-1} are polar reciprocals with respect to U_r; shew that U_{r+s} and U_{r-s} are also polar reciprocals with respect to U_r.

11. The rectangles under opposite edges of a tetrahedron are the same whichever pair is taken; prove that the straight lines joining its corners to the corners of the polar tetrahedron with respect to the circumscribed sphere will meet in a point.

12. If four of the eight common tangent planes of three conicoids meet in a point, the other four will also meet in a point.

13. A plane moves so that the sum of the squares of its distances from two of the angles of a tetrahedron is equal to the sum of the squares of its distances from the other two; prove that its envelope is a hyperbolic paraboloid cutting the faces of the tetrahedron in hyperbolas each having its asymptotes passing through two of the angles of the tetrahedron.

14. If $ABCD$ be a tetrahedron, self-conjugate with respect to a paraboloid, and DA, DB, DC meet the surface in A_1, B_1, C_1 respectively; shew that

$$\left|\overline{\frac{DA_1}{AA_1}}\right|^2 + \left|\overline{\frac{DB_1}{BB_1}}\right|^2 + \left|\overline{\frac{DC_1}{CC_1}}\right|^2 = 1.$$

15. If a tetrahedron have a self-conjugate sphere, and if its radius be R, prove that $\frac{1}{6R^2} = \Sigma \frac{1}{2S - 3s}$, where s is the sum of the squares of the edges of one face, and S the sum of the squares of all the edges.

16. Shew that the locus of the centres of all conicoids which circumscribe a quadrilateral is a straight line.

17. The locus of the pole of a fixed plane with respect to the conicoids which circumscribe a quadrilateral is a straight line.

18. The polar plane of a fixed point with respect to any conicoid which circumscribes a given quadrilateral passes through a fixed line.

19. The sides of a twisted quadrilateral touch a conicoid; shew that the four points of contact lie on a plane.

20. A system of conicoids circumscribes a quadrilateral: shew (1) that one conicoid of the system will pass through a given point, (2) that two of the conicoids will touch a given line, (3) that one conicoid will touch a given plane. Shew also that the conicoids are cut in involution by any straight line; also that the pairs of tangent planes through any line are in involution.

21. If three conicoids have a common self-polar tetrahedron, the twenty-four tangent planes at their eight common points touch a conicoid, and the twenty-four points of contact of their eight common tangent planes lie on another conicoid.

22. Nine conicoids have a common self-polar tetrahedron; shew that the eight points of intersection of any three, the eight points of intersection of any other three, and the eight points of intersection of the remaining three are all on a conicoid.

23. The sphere which circumscribes a tetrahedron self-polar with respect to a conicoid cuts the director-sphere orthogonally.

24. The feet of the perpendiculars from any point of the surface $\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}+\frac{d}{\delta}=0$, on the faces of the fundamental tetrahedron lie in a plane, a, b, c, d being proportional to the volumes of the tetrahedron formed by the centre of the inscribed sphere and the feet of the perpendiculars from it on any three of the faces, and the co-ordinates being quadriplanar.

25. The middle points of the twenty-eight lines which join two and two the centres of the eight spheres inscribed in any tetrahedron are on a cubic surface which contains the edges of the tetrahedron. Shew also that the feet of the perpendiculars from any point of the cubic surface on the faces of the tetrahedron lie on a plane.

26. The six edges of a tetrahedron are tangents to a conicoid. The plane through the three points of contact of the three edges which meet in the same vertex meet the face opposite to that vertex in a straight line: shew that the four such lines are generators of the same system of an hyperboloid.

27. When a tetrahedron is inscribed in a surface of the second degree, the tangent planes at its vertices meet the opposite faces in four lines which are generators of an hyperboloid.

28. The lines which join the vertices of a tetrahedron to the points of contact of any inscribed conicoid with the opposite faces are generators of an hyperboloid.

29. The lines which join the angular points of a tetrahedron to the angular points of the polar tetrahedron are generators of the same system of a conicoid.

30. Cones are described whose vertices are the vertices of a tetrahedron and bases the intersection of a conicoid with the opposite faces. The other planes of intersection of the cones and conicoid are produced to intersect the corresponding faces of the tetrahedron. Prove that the four lines of intersection are generating lines, of the same system, of a hyperboloid.

CHAPTER X.

SURFACES IN GENERAL.

204. We shall in the present Chapter discuss some properties of surfaces of higher degree than the second.

205. Let $F(x, y, z) = 0$ be the equation of any surface.

To find the points of intersection of the surface and the straight line whose equations are

$$\frac{x-x'}{l} = \frac{y-y'}{m} = \frac{z-z'}{n} = r,$$

we have the equation

$$F(x'+lr, \quad y'+mr, \quad z'+nr) = 0,$$

or

$$F(x', y', z') + r\left(l\frac{dF}{dx'} + m\frac{dF}{dy'} + n\frac{dF}{dz'}\right)$$
$$+ \frac{r^2}{1\,.\,2}\left(l\frac{d}{dx'} + m\frac{d}{dy'} + n\frac{d}{dz'}\right)^2 F + \ldots\ldots = 0 \ldots \text{(i)}.$$

If the equation of the surface be of the n^{th} degree, the equation (i) will be of the n^{th} degree. Hence a straight line will meet a surface of the n^{th} degree in n points, and any plane will cut the surface in a curve of the n^{th} degree.

206. *To find the equation of the tangent plane at any point of a surface.*

If (x', y', z') be a point on $F(x, y, z) = 0$, one root of the equation for r, found in the preceding article, will be zero.

Two roots will be zero if l, m, n satisfy the relation

$$l\frac{dF}{dx'}+m\frac{dF}{dy'}+n\frac{dF}{dz'}=0.$$

The line will in that case be a tangent line to the surface; and the locus of all the tangent lines is found by eliminating l, m, n by means of the equations of the straight line. We thus obtain the required equation of the tangent plane

$$(x-x')\frac{dF}{dx'}+(y-y')\frac{dF}{dy'}+(z-z')\frac{dF}{dz'}=0.$$

If the equation of the surface be $z-f(x, y)=0$, it is easy to deduce from the above, or to shew independently, that the equation of the tangent plane at (x', y', z') is

$$z-z'=(x-x')\frac{df}{dx'}+(y-y')\frac{df}{dy'}.$$

207. The two real or imaginary lines whose direction-cosines satisfy both the relations

$$l\frac{dF}{dx'}+m\frac{dF}{dy'}+n\frac{dF}{dz'}=0,$$

and $$\left(l\frac{d}{dx'}+m\frac{d}{dy'}+n\frac{d}{dz'}\right)^2F=0,$$

meet the surface in *three* coincident points.

Hence at any point of a surface two real or imaginary tangent lines meet the surface in three coincident points. These are called the *inflexional tangents.*

208. The tangent plane at any point of a surface will meet the surface in a curve of the n^{th} degree; and, since every line which is in the tangent plane, and which passes through its point of contact, meets the surface, and therefore the curve of intersection, in two points, it follows that the point of contact is a singular point in the curve of intersection.

When the inflexional tangents are imaginary, the point is a conjugate point on the curve of intersection. When the inflexional tangents are real, two branches of the curve of intersection pass through the point of contact; and these branches coincide when the inflexional tangents are coincident.

209. *The section of any surface by a plane parallel and indefinitely near the tangent plane at any point is a conic.*

Let any point on a surface be taken for origin, and let the tangent plane at the point be the plane $z=0$. Let the equation of the surface be $z=f(x, y)$; then, since $z=0$ is the tangent plane at the origin, we have

$$z = ax^2 + 2hxy + by^2$$

+ higher powers of the variables.

Hence, if we only consider points so near the origin that we may neglect the third and higher powers of the co-ordinates, the section of the given surface by the plane $z=k$, is the same as the section of the conicoid whose equation is

$$z = ax^2 + by^2 + 2hxy,$$

by the plane $z=k$; the section is therefore a conic.

The conic in which a surface is cut by a plane parallel and indefinitely near the tangent plane at any point, is called the *indicatrix* at the point; and points on a surface are said to be *elliptic, parabolic,* or *hyperbolic,* according as the indicatrix is an ellipse, parabola, or hyperbola.

210. If, at the point (x', y', z') on the surface $F(x, y, z)=0$, we have

$$\frac{dF}{dx'} = \frac{dF}{dy'} = \frac{dF}{dz'} = 0,$$

every straight line through the point (x', y', z') will meet the surface in two coincident points.

Such a point is called a *singular point* on the surface. All straight lines whose direction-cosines satisfy the relation

$$\left(l\frac{d}{dx'} + m\frac{d}{dy'} + n\frac{d}{dz'}\right)^2 F = 0,$$

will meet the surface in three coincident points and are called tangent lines. Eliminating l, m, n, by means of the equations of the line, we obtain the locus of all the tangent lines, viz. the cone whose equation is

$$(x-x')^2\frac{d^2F}{dx'^2}+(y-y')^2\frac{d^2F}{dy'^2}+(z-z')^2\frac{d^2F}{dz'^2}$$
$$+2(y-y')(z-z')\frac{d^2F}{dy'dz'}+2(z-z')(x-x')\frac{d^2F}{dz'dx'}$$
$$+2(x-x')(y-y')\frac{d^2F}{dx'dy'}=0.$$

When the tangent lines at any point of a surface form a cone, the point is called a *conical point;* and when all the tangent lines lie in one or other of two planes, the point is called a *nodal point.*

Ex. 1. Find the equation of the tangent plane at any point of the surface $x^{\frac{2}{3}}+y^{\frac{2}{3}}+z^{\frac{2}{3}}=a^{\frac{2}{3}}$; and shew that the sum of the squares of the intercepts on the axes, made by a tangent plane, is constant. $=a^2$

Ex. 2. Prove that the tetrahedron formed by the co-ordinate planes, and any tangent plane of the surface $xyz=a^3$, is of constant volume.

Ex. 3. Find the co-ordinates of the conical points on the surface $xyz-a(x^2+y^2+z^2)+4a^3=0$; and shew that the tangent cones at the conical points are right circular.

[The conical points are $(2a, 2a, 2a,)$ $(2a, -2a, -2a,)$ $(-2a, 2a, -2a)$ and $(-2a, -2a, 2a)$. The tangent cone at the first point is

$$x^2+y^2+z^2-2yz-2zx-2xy=0.]$$

Envelopes.

211. *To find the locus of the ultimate intersections of a series of surfaces, whose equations involve one arbitrary parameter.*

Let the equation of one of the surfaces be

$$F(x, y, z, a)=0,$$

where a is the parameter.

A consecutive surface is given by the equation

$$F(x, y, z, a + \delta a) = 0,$$

or $$F(x, y, z, a) + \frac{d}{da} F(x, y, z, a)\, \delta a + \ldots\ldots = 0.$$

Hence, when δa is made indefinitely small, we have for the ultimate intersection of the two surfaces the curve given by the equations

$$F(x, y, z, a) = 0, \text{ and } \frac{d}{da} F(x, y, z, a) = 0.$$

The required envelope is found by eliminating a from these equations.

The curve in which any surface is met by the consecutive surface is called the *characteristic* of the envelope. Every characteristic will meet the next in one or more points, and the locus of these points is called the *edge of regression* or *cuspidal edge* of the envelope.

212. *To find the equations of the edge of regression of the envelope.*

The equations of the characteristic corresponding to the surface $F(x, y, z, a) = 0$ are

$$F(x, y, z, a) = 0 \text{ and } \frac{d}{da} F(x, y, z, a) = 0.$$

The equations of the next consecutive characteristic are therefore

$$F(x, y, z, a + \delta a) = 0 \text{ and } \frac{d}{da} F(x, y, z, a + \delta a) = 0,$$

or $$F + \frac{dF}{da} \delta a + \ldots = 0, \text{ and } \frac{dF}{da} + \frac{d^2F}{da^2} \delta a + \ldots\ldots\ldots\ldots = 0.$$

Hence at any point of the edge of regression we must have

$$F = 0,\ \frac{dF}{da} = 0, \text{ and } \frac{d^2F}{da^2} = 0.$$

The equations of the edge are found by eliminating a from the above equations.

213. *The envelope of a system of surfaces, whose equation involves only one parameter, will touch each of the surfaces along a curve.*

Let A, B, C be three consecutive surfaces of the system; and let PQ be the curve of intersection of the surfaces A and B, and $P'Q'$ the curve of intersection of the surfaces B and C. Then the curves PQ and $P'Q'$ are ultimately on the envelope. Let R be any point on the curve PQ; and let S, T be two points, very near the point R, one on the curve PQ, and the other on $P'Q'$. Then the plane RST will in the limiting position be the tangent plane at R both to the surface B and to the envelope; and hence the envelope touches the surface B, and similarly every other surface of the system, along a curve.

214. *To find the envelope of a series of surfaces whose equations involve two arbitrary parameters.*

Let the equation of any surface of the system be

$$F(x, y, z, a, b) = 0,$$

where a, b are the parameters.

A consecutive surface of the system is

$$F(x, y, z, a + \delta a, b + \delta b) = 0,$$

or $$F(x, y, z, a, b) + \delta a \frac{dF}{da} + \delta b \frac{dF}{db} + \dots\dots\dots = 0.$$

Hence, when δa and δb are made indefinitely small, we must have at a point of ultimate intersection

$$F = 0, \text{ and } \delta a \frac{dF}{da} + \delta b \frac{dF}{db} = 0,$$

or, since δa and δb are independent,

$$F = 0, \frac{dF}{da} = 0, \text{ and } \frac{dF}{db} = 0.$$

Hence the curve of intersection of F with any surface consecutive to it goes through the point which satisfies the equations

$$F = 0, \ \frac{dF}{da} = 0, \text{ and } \frac{dF}{db} = 0.$$

The required envelope is found by eliminating a and b from the above equations.

215. *To shew that the envelope of a series of surfaces, whose equations involve two arbitrary parameters, touches each surface of the series.*

Let the curves of intersection of the surface F with consecutive surfaces of the system pass through the point P; then P is a point on the envelope. Let F_1, F_2 be any two surfaces consecutive to F, and let Q, R be the points on the envelope which correspond to these surfaces. Then all surfaces consecutive to F_1, and therefore the surface F, will pass through Q; similarly the surface F will pass through R. Hence, in the limit, the envelope and the surface F have the three points P, Q, R, which are indefinitely near to one another, in common; they therefore have a common tangent plane. Hence the envelope *touches* the surface F, and similarly for any other surface.

Ex. 1. Find the envelope of the plane which forms with the co-ordinate planes a tetrahedron of constant volume. *Ans.* $xyz = \text{constant}$.

Ex. 2. Find the envelope of a plane such that the sum of the squares of its intercepts on the axes is constant. *Ans.* $x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = \text{constant}$.

Ex. 3. Find the equations of the edge of regression of the envelope of the plane $x \sin\theta - y\cos\theta = a\theta - cz$. *Ans.* $x^2 + y^2 = a^2$, $y = x\tan\dfrac{cz}{a}$.

FAMILIES OF SURFACES.

216. *To find the general functional and differential equations of conical surfaces.*

The equation of any cone, when referred to its vertex as origin, is homogeneous; and is therefore of the form

$$F\left(\frac{x}{z}, \ \frac{y}{z}\right) = 0.$$

Hence the equation of any cone whose vertex is at the point (α, β, γ) is of the form

$$F\left(\frac{x-\alpha}{z-\gamma}, \frac{y-\beta}{z-\gamma}\right)=0 \text{..............(i).}$$

This is the required functional equation.

The tangent plane at any point of a cone passes through the vertex of the cone. Hence, if the equation $F(x, y, z)=0$ represent a cone whose vertex is (α, β, γ), we have

$$(x-\alpha)\frac{dF}{dx}+(y-\beta)\frac{dF}{dy}+(z-\gamma)\frac{dF}{dz}=0 \text{......(ii),}$$

which is the required differential equation.

217. *To find the general functional and differential equations of cylindrical surfaces.*

A cylinder is the surface generated by a straight line which is always parallel to a given straight line, and which obeys some other law.

Let the equations of the fixed straight line be

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}.$$

The equations of any parallel line are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z}{n} \text{.....................(i),}$$

the two constants α and β being arbitrary.

Now, in order that the line (i) may generate a *surface*, there must be some relation between the constants α and β. Let this relation be expressed by the equation $\alpha=f(\beta)$; then, we have from (i)

$$x-\frac{l}{n}z=f\left(y-\frac{m}{n}z\right),$$

or $$F(nx-lz, ny-mz)=0 \text{ ....(ii),}$$

which is the required functional equation.

The tangent plane at any point of a cylinder is parallel to the axis of the cylinder. Hence, if the equation $F(x, y, z) = 0$ represent a cylinder, whose axis is parallel to the line

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n},$$

we have
$$l\frac{dF}{dx} + m\frac{dF}{dy} + n\frac{dF}{dz} = 0,$$

which is the required differential equation.

218. *To find the general functional and differential equations of conoidal surfaces.*

Def. A *conoidal* surface is a surface generated by the motion of a straight line which always meets a fixed straight line, is parallel to a fixed plane, and obeys some other law. The surface is called a *right* conoid when the fixed plane is perpendicular to the fixed line.

Let the fixed straight line be the line of intersection of the planes

$$lx + my + nz + p = 0,\ l'x + m'y + n'z + p' = 0;$$

and let the fixed plane, to which the moving line is to be parallel, be

$$\lambda x + \mu y + \nu z = 0.$$

The equations of any line which satisfies the given conditions are

$$lx + my + nz + p + A\,(l'x + m'y + n'z + p') = 0,$$

and
$$\lambda x + \mu y + \nu z + B = 0.$$

In order that the straight line may generate a surface, there must be some relation between the constants A and B. Let this relation be expressed by the equation $A = f(B)$; then we have

$$\frac{lx + my + nz + p}{l'x + m'y + n'z + p'} = f\,(\lambda x + \mu y + \nu z) \text{ .... (i)},$$

the required functional equation.

If we take two of the co-ordinate planes through the fixed straight line, and the third co-ordinate plane parallel to the fixed plane, the above equation reduces to the simple form

$$\frac{x}{y} = f(z) \text{......................} \text{(ii).}$$

The differential equation of conoidal surfaces which corresponds to the functional equation (ii), can be readily shewn to be

$$x\frac{dF}{dx} + y\frac{dF}{dy} = 0.$$

The differential equation may also be obtained as follows.

The generator through any point is a tangent line to the surface; and the condition that

$$\frac{\xi}{x} = \frac{\eta}{y} = \frac{\zeta - z}{0},$$

may be on the plane

$$(\xi - x)\frac{dF}{dx} + (\eta - y)\frac{dF}{dy} + (\zeta - z)\frac{dF}{dz} = 0$$

is

$$x\frac{dF}{dx} + y\frac{dF}{dy} = 0.$$

Ex. 1. Shew that $xyz = c(x^2 - y^2)$ represents a conoidal surface.

Ex. 2. Find the equation of the right conoid whose axis is the axis of z, and whose generators pass through the circle $x = a$, $y^2 + z^2 = b^2$.

Ans. $a^2y^2 + x^2z^2 = b^2x^2$.

Ex. 3. Find the equation of the right conoid whose axis is the axis of z, and whose generators pass through the curve given by the equations $x = a\cos nz$, $y = a\sin nz$. *Ans.* $y = x\tan nz$.

Ex. 4. Shew that the only conoid of the second degree is a hyperbolic paraboloid.

219. Cones, cylinders and conoids are special forms of *ruled* surfaces. There are two distinct classes of ruled surfaces, namely those on which consecutive generators intersect, and those on which consecutive generators do not intersect; these are called developable and skew surfaces respectively. We proceed to consider some properties of developable and skew surfaces.

220. Suppose we have any number of generating lines of a developable surface, that is any number of straight lines such that each intersects the next consecutive. Then, the plane containing the first two lines can be turned about the second line until it coincides with the plane containing the second and third lines; this plane can then be turned about the third line until it coincides with the plane through the third and fourth lines; and so on. In this way the whole surface can be developed into one plane without tearing.

221. The tangent plane at any point of a ruled surface must contain the generator through the point [Art. 129]. If the surface be a skew surface, the tangent plane will be different at different points of the same generator; but, if the surface be a developable surface, the tangent plane will be the same at all the different points of a given generator, for the tangent plane is the limiting position of the plane through the given generator and the next consecutive generator.

Since any tangent plane to a developable surface touches the surface at all points of a straight line, it follows from Art. 213, that a developable surface is the envelope of a plane whose equation contains only *one* variable parameter.

222. *To find the general differential equation of developable surfaces.*

The tangent plane at any point of a developable surface meets the surface in two consecutive generating lines which are the two inflexional tangents at the point.

Hence, at any point of a developable surface, the two lines given by the equations

$$l\frac{dF}{dx}+m\frac{dF}{dy}+n\frac{dF}{dz}=0,$$

and $$\left(l\frac{d}{dx}+m\frac{d}{dy}+n\frac{d}{dz}\right)^2F=0,$$

must coincide.

The condition that this may be the case is

$$\begin{vmatrix} \dfrac{d^2F}{dx^2}, & \dfrac{d^2F}{dxdy}, & \dfrac{d^2F}{dxdz}, & \dfrac{dF}{dx} \\ \dfrac{d^2F}{dxdy}, & \dfrac{d^2F}{dy^2}, & \dfrac{d^2F}{dydz}, & \dfrac{dF}{dy} \\ \dfrac{d^2F}{dxdz}, & \dfrac{d^2F}{dydz}, & \dfrac{d^2F}{dz^2}, & \dfrac{dF}{dz} \\ \dfrac{dF}{dx}, & \dfrac{dF}{dy}, & \dfrac{dF}{dz}, & 0 \end{vmatrix} = 0 \ldots\ldots \text{(i)}.$$

This is the required differential equation.

The differential equation may also be obtained from the property, proved in the last Article, that a developable surface is the envelope of a plane whose equation involves only *one* parameter.

For, the general equation of the tangent plane of a surface at the point (x, y, z) is

$$\zeta - z = (\xi - x)\frac{df}{dx} + (\eta - y)\frac{df}{dy}.$$

Hence, if the surface is a developable surface, there must be some relation connecting $\frac{df}{dx}$ and $\frac{df}{dy}$; that is, connecting $\frac{dz}{dx}$ and $\frac{dz}{dy}$; we therefore have

$$\frac{dz}{dx} = F\left(\frac{dz}{dy}\right).$$

Therefore

$$\frac{d^2z}{dx^2} = F'\left(\frac{dz}{dy}\right) \cdot \frac{d^2z}{dxdy},$$

and

$$\frac{d^2z}{dxdy} = F'\left(\frac{dz}{dy}\right) \cdot \frac{d^2z}{dy^2}.$$

Hence

$$\frac{d^2z}{dx^2} \cdot \frac{d^2z}{dy^2} = \left(\frac{d^2z}{dxdy}\right)^2,$$

which is equivalent to (i).

223. We can find the equation of the developable surface which passes through two given curves, in the following manner. The plane through any two consecutive generating lines of the surface will pass through two consecutive points on each of the given curves; hence the tangent plane to the required developable surface will *touch* each of the given curves.

Now the equation of a plane in its most general form contains *three* arbitrary constants, and the conditions of tangency of the two given curves will enable us to express any two of these constants in terms of the third, and the equation of the plane will thus be found in a form involving only *one* arbitrary parameter. The developable surface is then obtained as the envelope of the moving plane.

Ex. Find the equation of the developable surface whose generating lines pass through the two curves

$$y^2=4ax,\ z=0 \text{ and } x^2=4ay,\ z=c;$$

and shew that its edge of regression is given by the equations

$$cx^2-3ayz=0=cy^2-3ax(c-z).$$

Let one of the tangent planes of the developable be $lx+my+nz+1=0$. The plane touches the first curve, if $lx+my+1=0$ touches $y^2-4ax=0$; that is, if $l=am^2$. The plane touches the second curve, if $lx+my+nc+1=0$ touches $x^2=4ay$; that is, if $m(nc+1)=al^2$. Hence, the equation of the tangent plane of the developable is found in the form

$$am^2x+my+(a^3m^3-1)\frac{z}{c}+1=0 \text{ ....(i).}$$

The surface is therefore given by the elimination of m between (i), and

$$2amx+y+3\frac{a^3m^2z}{c}=0 \text{ ....(ii).}$$

For points on the edge of regression we have also

$$ax+3\frac{a^3mz}{c}=0 \text{..................................(iii).}$$

From (ii) and (iii) we have $m=-\dfrac{y}{ax}$; and therefore, from (iii), $cx^2=3ayz$. This is the equation of one surface through the edge of regression. We obtain another surface through the edge by substituting $m=-\dfrac{y}{ax}$ in (i); the result is $y^3z=x^3(c-z)$, and at all points common to the surfaces $cx^2=3ayz$, and $y^3z=x^3(c-z)$, we must have $cy^2=3ax(c-z)$.

224. *To shew that a conicoid can be drawn which will touch any skew surface along a generating line.*

Let AB, $A'B'$, $A''B''$ be three consecutive generators of any skew surface. Then, [Art. 134], a conicoid will have these three lines as generators of one system, and any line which intersects the three given lines will be a generator of the opposite system of the same conicoid. Through any point Q on $A'B'$ draw the line PQR to intersect the lines AB and $A''B''$. Then this line passes through three consecutive points of the given surface, and is therefore a tangent line to the surface. Hence the plane through $A'B'$ and PQR touches both the given surface and the conicoid. Hence the conicoid touches the given surface at all points of the line $A'B'$.

By means of the above theorem many properties of a ruled conicoid may be shewn to be true of all skew surfaces.

225. *To find the lines of striction of any skew surface.*

Def. The locus of the point on a generator of a ruled surface where it is met by the shortest distance between it and the next consecutive generator, is called the *line of striction* of the surface.

If we know the equations of any generating line, we can at once find the direction of the shortest distance between it and the next consecutive generator, and this shortest distance is a tangent line of the surface. Hence, in order to find the point on the line of striction, which corresponds to any particular generator, we have only to write down the condition that the normal at a point on the generator may be perpendicular to the shortest distance between the given generator and the next consecutive.

Ex. 1. To find the lines of striction of the hyperboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$$

The direction-cosines of a generator, and of the next consecutive generator, are proportional respectively to

$$a\sin\theta,\ -b\cos\theta,\ c,\ \text{and}\ a\sin(\theta+d\theta),\ -b\cos(\theta+d\theta),\ c.$$

Hence the direction-cosines of the shortest distance are proportional to

$$-bc\sin\theta,\ ca\cos\theta,\ ab.$$

Now, if (x, y, z) be the point where the shortest distance meets the consecutive generators, the normal at (x, y, z) must be perpendicular to the given generator, and also to the shortest distance. We therefore have

$$\frac{x}{a}\sin\theta - \frac{y}{b}\cos\theta - \frac{z}{c} = 0,$$

and $$\frac{x}{a^3}\sin\theta - \frac{y}{b^3}\cos\theta + \frac{z}{c^3} = 0.$$

Eliminating θ, we get for the lines of striction the intersection of the surface and the quartic

$$\frac{a^2}{x^2}\left(\frac{1}{b^2}+\frac{1}{c^2}\right)^2 + \frac{b^2}{y^2}\left(\frac{1}{c^2}+\frac{1}{a^2}\right)^2 = \frac{c^2}{z^2}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)^2.$$

Ex. 2. To find the lines of striction of the paraboloid whose equation is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2z.$$

All the generating lines of one system are parallel to the plane

$$\frac{x}{a} - \frac{y}{b} = 0 \quad \text{..................................(i).}$$

The shortest distance between two consecutive generators of this system will therefore be perpendicular to the plane (i). Hence, at a point on the corresponding line of striction, the normal to the surface is parallel to (i). The equations of the normal at (x, y, z) are

$$\frac{\xi - x}{\dfrac{x}{a^2}} = \frac{\eta - y}{-\dfrac{y}{b^2}} = \frac{\zeta - z}{-1}.$$

Hence one line of striction is the intersection of the surface and the plane

$$\frac{x}{a^3} + \frac{y}{b^3} = 0.$$

Similarly, the line of striction of the generators which are parallel to the plane $\frac{x}{a} + \frac{y}{b} = 0$ is the parabola in which the plane $\frac{x}{a^3} - \frac{y}{b^3} = 0$ cuts the surface.

[See a paper by Prof. Larmor, *Quarterly Journal of Mathematics*, Vol. XIX. page 381.]

226. *To find the general functional and differential equations of surfaces of revolution.*

Let the equations of the axis of revolution be

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}.$$

The equations of a section of the surface by a plane perpendicular to the axis are of the form

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2,$$

and $$lx+my+nz=p.$$

Hence, since there must be some relation between r^2 and p, the required functional equation is

$$(x-a)^2+(y-b)^2+(z-c)^2=f(lx+my+nz).$$

The normal at every point of a surface of revolution intersects the axis. The equations of the normal at the point (x', y', z') of the surface $F(x, y, z)=0$ are

$$\frac{x-x'}{\dfrac{dF}{dx'}}=\frac{y-y'}{\dfrac{dF}{dy'}}=\frac{z-z'}{\dfrac{dF}{dz'}}.$$

By writing down the condition that the normal may intersect the axis, we see that at every point of the surface,

$$\begin{vmatrix} \dfrac{dF}{dx}, & \dfrac{dF}{dy}, & \dfrac{dF}{dz} \\ x-a, & y-b, & z-c \\ l, & m, & n \end{vmatrix}=0;$$

this is the differential equation of surfaces of revolution.

NOTE. In the above, and also in Articles 216 and 217, we have obtained the functional equation and the differential equation by independent methods. The differential equation could however in each case be obtained from the functional equation; this we leave as an exercise for the student.

For fuller treatment of Families of Surfaces the student is referred to Salmon's *Solid Geometry*, Chapter XIII.

Examples on Chapter X.

1. Prove that a surface of the fourth degree can be described to pass through all the edges of a parallelopiped, and that if it pass through the centre it also passes through the diagonals of the figure.

2. Shew that at any point on the axis of z there are two tangent planes to the surface $a^2y^2 = x^2(c^2 - z^2)$.

3. Find the developable surface which passes through a parabola and the circle described in a perpendicular plane on the latus rectum as diameter.

4. Find the equation of the developable surface which contains the two curves

$$y^2 = 4ax,\ z = 0; \text{ and } (y - b)^2 = 4cz,\ x = 0;$$

and shew that its cuspidal edge lies on the surface

$$(ax + by + cz)^2 = 3abx(y + b).$$

5. The developable surface which passes through the two circles whose equations are $x^2 + y^2 = a^2$, $z = 0$, and $x^2 + z^2 = c^2$, $y = 0$, passes also through the rectangular hyperbola whose equations are

$$z^2 - y^2 = \frac{a^2c^2}{a^2 - c^2} \text{ and } x = 0.$$

6. Prove that the surface

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)^2 - 3\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) - \frac{z^2}{c^2} + \tfrac{1}{4} = 0$$

has two conical points, and two singular tangent planes.

7. Explain what is meant by a nodal line on a surface, and find the conditions for such a line on the surface $\phi(x, y, z) = 0$.

There is a nodal line on the surface $z(x^2 + y^2) + 2axy = 0$; find it.

8. Give a general explanation of the form of the surface $z(x^2 + y^2) = 2kxy$. Shew that every tangent plane meets the surface in an ellipse whose projection on a plane perpendicular to the nodal line is a circle.

9. Examine the general form of the surface

$$xyz - a^2x - b^2y - c^2z + 2abc = 0,$$

and shew that it has a conical point. Shew also that each of the planes passing through the conical point and a pair of the intersections with the axes touches the surface along a straight line.

10. If a ruled surface be such that at any point of it a straight line can be drawn lying wholly on the surface and intersecting the axis of z, then at every point of the surface

$$x^2\frac{d^2z}{dx^2} + 2xy\frac{d^2z}{dx\,dy} + y^2\frac{d^2z}{dy^2} = 0.$$

11. Shew that the surface whose equation is determined by the elimination of θ between the equations

$$x\cos\theta + y\sin\theta = a,$$

$$x\sin\theta - y\cos\theta = \frac{a}{c}(c\theta - z),$$

is a developable surface, and find its edge of regression.

12. What family of surfaces is represented by the equation $z = \phi\left(\frac{y}{x}\right)$? Describe the form of the surface whose equation is $\sin^{-1}\frac{z}{c} = n\tan^{-1}\frac{y}{x}$. If $n = 2$, prove that through any point an infinite number of planes can be drawn, each of which shall cut the surface in a conic section.

13. At a point on the surface $(x - y)z^2 + ax(z + a) = 0$ there is in general only one generator, but at certain points there are two, which are at right angles.

14. Any tangent plane to the surface $a(x^2 + y^2) + xyz = 0$ meets it again in a conic whose projection on the plane of xy is a rectangular hyperbola.

15. Shew that tangent planes at points on a generator of the surface $yx^2 - a^2z = 0$ cut $x = 0$ in parallel straight lines.

16. Prove that the equation $x^3 + y^3 + z^3 - 3xyz = a^3$ represents a surface of revolution, and find the equation of the generating curve.

17. From any point perpendiculars are drawn to the generators of the surface $z(x^2 + y^2) - 2mxy = 0$; shew that the feet of the perpendiculars lie upon a plane ellipse.

18. Shew that all the normals to a skew surface, at points on a generator, lie on a hyperbolic paraboloid whose vertex is at the point where the generator meets the shortest distance between it and the next.

19. A generator PQ of the surface $xyz - k(x^2 + y^2) = 0$ meets the axis of z in P. Prove that the tangent plane at Q meets the surface in a hyperbola passing through P, and that as Q moves along the generator the tangent at P to the hyperbola generates a plane.

20. Prove that all tangent planes to an anchor-ring which pass through the centre of the ring cut the surface in two circles.

Also if a surface be generated by the revolution of any conic section about an axis in its own plane, prove that a double tangent plane cuts the surface in two conic sections.

21. Prove that a flexible inextensible surface in the form of a hyperboloid of revolution of one sheet, cut open along a generator, may be bent so that the circle in the principal plane becomes the axis, and the generators the generating lines of a conoid of uniform pitch inclined to the axis at a constant angle.

22. Prove that every cubic surface has twenty-seven lines and forty-five triple tangent planes real or imaginary, and that every cubic surface which has a double line is a ruled surface.

Discuss some properties of the surface whose equation is

$$y^3 + x^2z + yzw = 0.$$

23. Four tangent planes to any skew surface which are drawn through the same generator have their cross-ratio equal to that of their four points of contact.

24. Any plane through a generator of a skew surface is a tangent plane at some point P and a normal plane at some point P'; shew also that there is a point O on the generator such that the rectangle $OP \,.\, OP'$ is constant for all planes through it.

25. Shew that the wave-surface, whose equation is

$$\frac{a^2x^2}{x^2+y^2+z^2-a^2} + \frac{b^2y^2}{x^2+y^2+z^2-b^2} + \frac{c^2z^2}{x^2+y^2+z^2-c^2} = 0,$$

has four conical points, and four singular tangent planes.

CHAPTER XI.

CURVES.

227. WE have already seen that any two equations will represent a curve. By means of the two equations of the curve, we can, theoretically at any rate, express the three co-ordinates of any point as functions of a single variable; we may, for example, suppose the three co-ordinates of any point of a curve expressed as functions of the length of the arc measured along the curve from some fixed point.

228. *To find the equations of the tangent at any point of a curve.*

Let x, y, z be the co-ordinates of any point P on the curve, and let $x+\delta x$, $y+\delta y$, $z+\delta z$ be the co-ordinates of an adjacent point Q. Then, if δs be the length of the arc PQ, we have, since the arc is ultimately equal to the chord,

$$\delta x^2+\delta y^2+\delta z^2=\delta s^2;$$

$$\therefore \left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2+\left(\frac{dz}{ds}\right)^2=1.$$

Also, since the direction-cosines of the chord PQ are proportional to δx, δy, δz, and the tangent coincides with the ultimate position of the chord, the direction-cosines of the tangent are equal to

$$\frac{dx}{ds},\ \frac{dy}{ds},\ \frac{dz}{ds},$$

so that the required equations of the tangent at (x, y, z) are

$$\frac{\xi-x}{\frac{dx}{ds}}=\frac{\eta-y}{\frac{dy}{ds}}=\frac{\zeta-z}{\frac{dz}{ds}}.$$

If the curve be the curve of intersection of the two surfaces

$$F(x, y, z) = 0 \text{ and } G(x, y, z) = 0,$$

the tangent line at any point is the line of intersection of the tangent planes of the two surfaces at that point. Hence the equations of the tangent at any point (x, y, z) are

$$(\xi - x)\frac{dF}{dx} + (\eta - y)\frac{dF}{dy} + (\zeta - z)\frac{dF}{dz} = 0,$$

$$(\xi - x)\frac{dG}{dx} + (\eta - y)\frac{dG}{dy} + (\zeta - z)\frac{dG}{dz} = 0.$$

229. *To find on a given surface a curve such that the tangent line at any point makes a maximum angle with a given plane.*

It is clear that the tangent line to such a curve at any point is in the tangent plane to the surface at that point, and is perpendicular to the line of intersection of the tangent plane and the given plane.

Let the equation of the given plane be

$$lx + my + nz = 0.$$

Then the direction-cosines of the line of intersection of the given plane and the tangent plane at any point (x, y, z) of the surface $F(x, y, z) = 0$, are proportional to

$$m\frac{dF}{dz} - n\frac{dF}{dy},\quad n\frac{dF}{dx} - l\frac{dF}{dz},\quad l\frac{dF}{dy} - m\frac{dF}{dx}.$$

The direction-cosines of the tangent to the curve are

$$\frac{dx}{ds},\quad \frac{dy}{ds},\quad \frac{dz}{ds}.$$

Hence we have

$$\frac{dx}{ds}\left(m\frac{dF}{dz} - n\frac{dF}{dy}\right) + \frac{dy}{ds}\left(n\frac{dF}{dx} - l\frac{dF}{dz}\right)$$
$$+ \frac{dz}{ds}\left(l\frac{dF}{dy} - m\frac{dF}{dx}\right) = 0,$$

the required differential equation.

If the given plane be the plane $z=0$, the differential equation of a line of greatest slope will be

$$\frac{dF}{dx}\frac{dy}{ds}-\frac{dF}{dy}\frac{dx}{ds}=0.$$

Ex. Find the lines of greatest slope to the plane $z=0$ on the right conoid whose equation is $$x=yf(z).$$

The differential equation of the projection on $z=0$ of a line of greatest slope is $x\,dx+y\,dy=0$.

Hence the projections of the lines of greatest slope on the plane $z=0$ are circles.

230. *Definitions.* If A, B, C be three points on a curve, the limiting position of the plane ABC, when A, C are supposed to move up to and ultimately to coincide with B, is called the *osculating plane* at B.

The circle ABC in its limiting position is called the *circle of curvature* at B, the radius of the circle is the *radius of curvature*, and its centre the *centre of curvature* at B.

The normals to a curve at any point are all in the plane through the point perpendicular to the tangent to the curve: this plane is called the *normal plane* at the point.

The normal which is in the osculating plane at any point of a curve is called the *principal normal*.

The normal which is perpendicular to the osculating plane is called the *binormal*.

The surface which is the envelope of all the normal planes of a curve is called the *polar developable*.

The angle between the osculating planes at any two points P, Q of a curve is called the *whole torsion* of the arc PQ. The limiting value of the ratio of the whole torsion to the arc is called the *torsion at a point*.

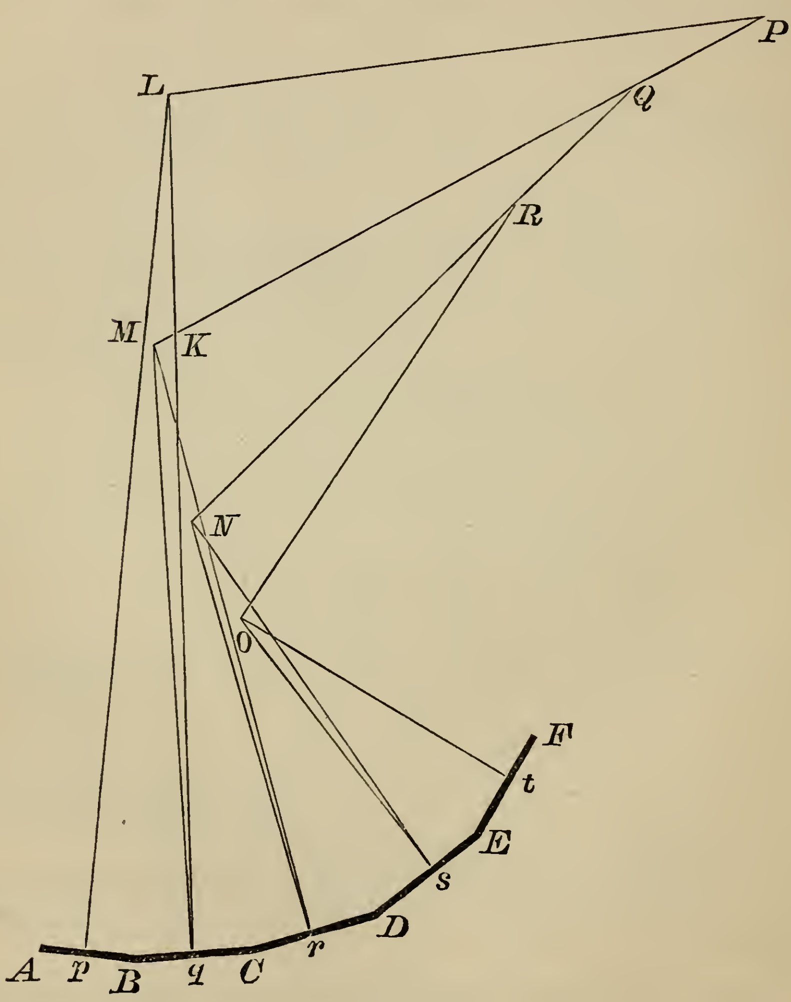
The radius of the circle whose curvature is equal to the torsion of the curve at any point, is called the *radius of torsion* at that point, and is represented by σ.

The radius of the sphere which passes through four consecutive points of a curve is called the *radius of spherical curvature*.

NOTE. In what follows we shall have frequent occasion to employ differential coefficients with respect to the arc; and we shall for shortness write x', x'', x''' &c. instead of

$$\frac{dx}{ds}, \ \frac{d^2x}{ds^2}, \ \frac{d^3x}{ds^3} \text{ \&c.}$$

231. In the annexed figure A, B, C, D, E, F... are supposed to be consecutive points of a curve, and p, q, r... are the middle points of the chords AB, BC, CD.... Planes are



drawn through p, q, r... perpendicular to the chords AB, BC, CD..., and LP, MQP, NRQ... are the lines of intersection of the planes through p and q, q and r, r and s,.... The lines pL, qL are in the plane ABC, and perpendicular respectively to AB and BC; the lines qM, rM are in the plane BCD, and perpendicular respectively to BC, CD.

Then, in the limit, when the chords AB, BC, CD... become indefinitely small the planes ABC, BCD,... become osculating planes of the curve; the planes pLP, qMQ,... become normal planes of the curve; the points L, M, N become centres of curvature of the curve; the lines LP, MQP, NRQ... become generating lines of the polar surface, and are called polar lines; and the points P, Q, R... become consecutive points on the edge of regression of the polar surface.

All points on the plane pLP are equidistant from A and B, all points on the plane qMP are equidistant from B and C, and all points on the plane rMP are equidistant from C and D; therefore a sphere with P for centre will pass through A, B, C, D; hence *the edge of regression of the polar surface is the locus of the centre of spherical curvature.*

232. *To find the equation of the osculating plane at any point of a curve.*

Let P, Q, R be three consecutive points on the curve such that $PQ = QR = \delta s$; and let s be the length of the arc measured from some fixed point up to Q.

Then, if the co-ordinates of Q be x, y, z, those of P, for which the arc is $s - \delta s$, will be, if we neglect powers of δs above the second,

$$x - x'\delta s + \frac{x''}{2}\delta s^2,\ y - y'\delta s + \frac{y''}{2}\delta s^2,\ z - z'\delta s + \frac{z''}{2}\delta s^2;$$

and the co-ordinates of R will be found by changing the sign of δs.

The equation of any plane through Q is of the form

$$L(\xi - x) + M(\eta - y) + N(\zeta - z) = 0.$$

If this plane pass through the points P and R, we must have

$$Lx' + My' + Nz' = 0,$$
$$Lx'' + My'' + Nz'' = 0;$$

and, eliminating L, M, N, we have the required equation of the osculating plane, namely

$$\begin{vmatrix} \xi - x, & \eta - y, & \zeta - z \\ x', & y', & z' \\ x'', & y'', & z'' \end{vmatrix} = 0.$$

233. *To find the equations of the principal normal, and the curvature, at any point of a curve.*

Let P, Q, R be three points on a curve such that

$$PQ = QR = \delta s.$$

Then, if V be the middle point of PR, QV is in the plane PQR; and, since the *chords* PQ and QR only differ by cubes of δs, QV is ultimately perpendicular to PR, and is therefore the principal normal at Q.

Then, the co-ordinates of P, Q, R being as in the last Article, the co-ordinates of V are

$$x + \frac{x''}{2}\delta s^2, \quad y + \frac{y''}{2}\delta s^2, \quad z + \frac{z''}{2}\delta s^2.$$

Hence the equations of QV are

$$\frac{\xi - x}{x''} = \frac{\eta - y}{y''} = \frac{\zeta - z}{z''} \text{..............} \quad \text{(i).}$$

Again, the circle PQR, in its limiting position, is the circle of curvature. Hence, if ρ be the radius of curvature, we have in the limit

$$2\rho = \frac{PQ^2}{QV}.$$

But $\qquad QV^2 = \frac{\delta s^4}{4}(x''^2 + y''^2 + z''^2)$, and $PQ = \delta s$;

$$\therefore \frac{1}{\rho^2} = x''^2 + y''^2 + z''^2.$$

Hence, the direction-cosines of the principal normal, which from (i) are proportional to x'', y'', z'', are equal to

$$\rho x'', \rho y'' \text{ and } \rho z''.$$

The co-ordinates of the centre of curvature are easily seen to be

$$x + \rho^2 x'',\ y + \rho^2 y'',\ z + \rho^2 z''.$$

234. *To find the direction-cosines of the binormal.*

The binormal is perpendicular to the osculating plane. Hence, if l, m, n be the direction-cosines of the binormal, we have from Art. 232

$$\frac{l}{y'z'' - z'y''} = \frac{m}{z'x'' - x'z''} = \frac{n}{x'y'' - y'x''}.$$

But

$$(y'z'' - z'y'')^2 + (z'x'' - x'z'')^2 + (x'y'' - y'x'')^2$$
$$= (x'^2 + y'^2 + z'^2)(x''^2 + y''^2 + z''^2) - (x'x'' + y'y'' + z'z'')^2$$
$$= \frac{1}{\rho^2}$$

since $$x'^2 + y'^2 + z'^2 = 1,$$

and therefore $$x'x'' + y'y'' + z'z'' = 0.$$

Hence the required direction-cosines are

$$\rho(y'z'' - z'y''),\ \rho(z'x'' - x'z''),\ \rho(x'y'' - y'x'').$$

235. *To find the measure of torsion at any point of a curve.*

Let l, m, n be the direction-cosines of the normal to the osculating plane at P; and let $l + \delta l$, $m + \delta m$, $n + \delta n$ be the direction-cosines of the normal to the osculating plane at Q, where $PQ = \delta s$. Then, if $\delta\tau$ be the angle between the osculating planes, we have

$$\sin^2 \delta\tau = (m\delta n - n\delta m)^2 + (n\delta l - l\delta n)^2 + (l\delta m - m\delta l)^2.$$

Hence, in the limit, we have

$$\left(\frac{d\tau}{ds}\right)^2 = \left(m\frac{dn}{ds} - n\frac{dm}{ds}\right)^2 + \left(n\frac{dl}{ds} - l\frac{dn}{ds}\right)^2 + \left(l\frac{dm}{ds} - m\frac{dl}{ds}\right)^2,$$

or, $$\frac{1}{\sigma^2} = (mn' - m'n)^2 + (nl' - n'l)^2 + (lm' - l'm)^2 \ldots\ldots \text{(i)}.$$

Now $$l = \rho\,(y'z'' - z'y'');$$

$$\therefore\ l' = \rho\,(y'z''' - z'y''') + \frac{d\rho}{ds}(y'z'' - z'y''),$$

and similarly for m' and n'.

Hence $$mn' - m'n = \rho^2\,(z'x'' - x'z'')\,(x'y''' - y'x''') - \rho^2\,(z'x''' - x'z''')\,(x'y'' - y'x'')$$

$$= \rho^2 x' \begin{vmatrix} x', & y', & z' \\ x'', & y'', & z'' \\ x''', & y''', & z''' \end{vmatrix}.$$

We can find similar expressions for $nl' - n'l$, and for $lm' - l'm$; and substituting in (i), we have

$$\frac{1}{\rho^2\sigma} = \begin{vmatrix} x', & y', & z' \\ x'', & y'', & z'' \\ x''', & y''', & z''' \end{vmatrix}.$$

236. *To find the condition that a curve may be a plane curve.*

Let x, y, z be the co-ordinates of any point P on the curve, expressed in terms of the arc measured from a fixed point up to P; and let Q be the point at a distance σ measured along the curve from P. Then the co-ordinates of Q will be

$$x + \sigma x' + \frac{\sigma^2}{\underline{|2}}\,x'' + \frac{\sigma^3}{\underline{|3}}\,x''' + \ldots\ldots,$$

$$y + \sigma y' + \frac{\sigma^2}{\underline{|2}}\,y'' + \frac{\sigma^3}{\underline{|3}}\,y''' + \ldots\ldots,$$

$$z + \sigma z' + \frac{\sigma^2}{\underline{|2}}\,z'' + \frac{\sigma^3}{\underline{|3}}\,z''' + \ldots\ldots

If all points of the curve are on the fixed plane

$$Ax + By + Cz + D = 0,$$

the equation

$$\begin{aligned} &A\left(x + \sigma x' + \frac{\sigma^2}{\lfloor 2} x'' + \frac{\sigma^3}{\lfloor 3} x''' + \dots\right) \\ &+ B\left(y + \sigma y' + \frac{\sigma^2}{\lfloor 2} y'' + \frac{\sigma^3}{\lfloor 3} y''' + \dots\right) \\ &+ C\left(z + \sigma z' + \frac{\sigma^2}{\lfloor 2} z'' + \frac{\sigma^3}{\lfloor 3} z''' + \dots\right) + D = 0, \end{aligned}$$

will be satisfied for *all values* of σ.

The coefficients of all the different powers of σ must therefore be zero. Hence we have

$$\begin{aligned} Ax' + By' + Cz' &= 0, \\ Ax'' + By'' + Cz'' &= 0, \\ Ax''' + By''' + Cz''' &= 0. \end{aligned}$$

The elimination of A, B, C gives

$$\begin{vmatrix} x', & y', & z' \\ x'', & y'', & z'' \\ x''', & y''', & z''' \end{vmatrix} = 0,$$

a relation which, since P is arbitrary, must be satisfied at all points of the given curve.

From the result of the preceding Article it will be seen that the above condition simply expresses the fact that the torsion is zero at all points of a plane curve.

The condition that a curve may be a plane curve may also be obtained in the following manner.

The direction-cosines of the normal to the osculating plane are [Art. 234]

$$\rho\,(y'z'' - z'y''),\ \rho\,(z'x'' - x'z'') \text{ and } \rho\,(x'y'' - y'x'').$$

Since these are constant, we have

$$\rho\,(y'z''' - z'y''') + \frac{d\rho}{ds}\,(y'z'' - z'y'') = 0,$$

$$\rho\,(z'x''' - x'z''') + \frac{d\rho}{ds}\,(z'x'' - x'z'') = 0,$$

and $$\rho\,(x'y''' - y'x''') + \frac{d\rho}{ds}\,(x'y'' - y'x'') = 0.$$

Multiply these equations in order by x'', y'', z'' and add: we then have

$$x''\,(y'z''' - z'y''') + y''\,(z'x''' - x'z''') + z''\,(x'y''' - y'x''') = 0,$$

which is the same condition as before.

237. *To find the centre and radius of spherical curvature.*

The locus of the centre of spherical curvature is the edge of regression of the polar surface, that is of the envelope of normal planes of the curve.

The equation of the normal plane at the point (x, y, z) is

$$(\xi - x)\,x' + (\eta - y)\,y' + (\zeta - z)\,z' = 0 \quad\text{.........}\quad \text{(i)}.$$

Hence [Art. 212] the corresponding point on the edge of regression is the point of intersection of (i), and the two planes

$$(\xi - x)\,x'' + (\eta - y)\,y'' + (\zeta - z)\,z''$$
$$= x'^2 + y'^2 + z'^2 = 1 \quad\text{............}\quad \text{(ii)},$$

and $$(\xi - x)\,x''' + (\eta - y)\,y''' + (\zeta - z)\,z''' = 0 \quad\text{...}\quad \text{(iii)},$$

since $$x'x'' + y'y'' + z'z'' = 0.$$

238. In the figure to Art. 231, we have

$$\rho = pL = qL, \quad \rho + \delta\rho = qM = rM,$$

and $$\delta\tau = LqM = LPM.$$

If K be the point of intersection of MQP and qKL, we have to the second order, $Mq = Kq$, and $KP = LP$;

$$\therefore\ LK = \delta\rho,$$

and $$LP = \frac{LK}{\delta\tau} = \frac{d\rho}{d\tau} \text{ ultimately} \quad\text{...........}\quad \text{(i)}.$$

Also $$pP^2 = pL^2 + LP^2;$$

$$\therefore\ R^2 = \rho^2 + \left(\frac{d\rho}{d\tau}\right)^2 \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)},$$

where R is the radius of spherical curvature.

Projecting the sides of the triangle KLP on the axis of x, we have, if l, m, n be the direction-cosines of the binormal,

$$\delta\rho \,.\, \rho x'' + \frac{d\rho}{d\tau}\, l - \frac{d\rho}{d\tau}(l + \delta l) = 0;$$

therefore ultimately $$\rho x'' = \frac{d\rho}{d\tau} \cdot \frac{dl}{d\rho} = \frac{dl}{ds}\frac{ds}{d\tau},$$

or $$\rho x'' = \sigma l' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}.$$

Since $l = \rho(y'z'' - z'y'')$ [Art. 234] we have from (iii)

$$\rho x'' = \sigma\rho(y'z''' - z'y''') + \sigma\frac{d\rho}{ds}(y'z'' - z'y'').$$

Similarly $$\rho y'' = \sigma\rho(z'x''' - x'z''') + \sigma\frac{d\rho}{ds}(z'x'' - x'z''),$$

and $$\rho z'' = \sigma\rho(x'y''' - y'x''') + \sigma\frac{d\rho}{ds}(x'y'' - y'x'').$$

Multiply the last three equations by x'', y'', z'' respectively and add; then we have, as in Art. 235,

$$\frac{1}{\rho^2\sigma} = \begin{vmatrix} x', & y', & z' \\ x'', & y'', & z'' \\ x''', & y''', & z''' \end{vmatrix} \ldots\ldots\ldots\ldots\ldots\text{(iv)}.$$

239. Since, in the figure to Art. 231, M and L are the feet of the perpendiculars from q on two consecutive tangents to the curve PQR, if we substitute R, ρ and τ for r, p, ψ in either of the known formulae $r\dfrac{dr}{dp}$ or $p + \dfrac{d^2p}{d\psi^2}$ for the radius of curvature of a plane curve, we shall obtain the radius of curvature of the edge of regression.

Hence the radius of curvature of the edge of regression is equal to

$$R\frac{dR}{d\rho}, \text{ or to } \rho+\frac{d^2\rho}{d\tau^2}.$$

[For this and the preceding article see a paper by Dr Routh, *Quarterly Journal*, Vol. VII.]

240. The following examples will illustrate the use of the different formulae we have investigated in this chapter.

Ex. 1. *To find the curvature and the torsion of a helix.*

A helix is a curve traced on a right circular cylinder so as to cut all the generating lines at the same angle. Its equations are easily seen to be

$$x=a\cos\theta,\ y=a\sin\theta,\ z=a\theta\tan\alpha.$$

Hence $\qquad x'=-a\sin\theta\,.\,\theta',\ y'=a\cos\theta\,.\,\theta',\ z'=a\tan\alpha\,.\,\theta'.$

Square and add, then $1=a^2\theta'^2\sec^2\alpha$.

We therefore have $\quad x''=-\cos\theta\dfrac{\cos^2\alpha}{a},\ y''=-\sin\theta\dfrac{\cos^2\alpha}{a},\ z''=0;$

and also $\qquad x'''=\dfrac{1}{a^2}\sin\theta\cos^3\alpha,\ y'''=-\dfrac{1}{a^2}\cos\theta\cos^3\alpha,\ z'''=0.$

Hence $\qquad \dfrac{1}{\rho^2}=\dfrac{\cos^4\alpha}{a^2},\text{ or }\rho=\dfrac{a}{\cos^2\alpha};$

and
$$\frac{1}{\rho^2\sigma}=\begin{vmatrix} -\sin\theta\cos\alpha, & \cos\theta\cos\alpha, & \sin\alpha \\ -\frac{1}{a}\cos\theta\cos^2\alpha, & -\frac{1}{a}\sin\theta\cos^2\alpha, & 0 \\ \frac{1}{a^2}\sin\theta\cos^3\alpha, & -\frac{1}{a^2}\cos\theta\cos^3\alpha, & 0 \end{vmatrix}$$

$$=\frac{1}{a^3}\cos^5\alpha\sin\alpha;$$

$$\therefore\ \sigma=\frac{a}{\sin\alpha\cos\alpha}.$$

It should be noticed that the principal normals all intersect perpendicularly the axis of the cylinder. This is seen at once by writing down the equations of the principal normal at θ, namely

$$\frac{x-a\cos\theta}{\cos\theta}=\frac{y-a\sin\theta}{\sin\theta}=\frac{z-a\theta\tan\alpha}{0}.$$

Ex. 2. To find the equations of the principal normal, and of the osculating plane at any point of the curve given by the equations

$$x=4a\cos^3\theta,\ y=4a\sin^3\theta,\ z=3c\cos 2\theta.$$

We have

$$x'=-12a\cos^2\theta\sin\theta\,.\,\theta',$$
$$y'=\ \ 12a\sin^2\theta\cos\theta\,.\,\theta',$$
$$z'=-6c\sin 2\theta\,.\,\theta'.$$

Square and add, then $1=6\sqrt{(a^2+c^2)}\sin 2\theta\,.\,\theta'$.

Hence $$x'=-\frac{a}{\sqrt{(a^2+c^2)}}\cos\theta,\ y'=\frac{a}{\sqrt{(a^2+c^2)}}\sin\theta,\ z'=-\frac{c}{\sqrt{(a^2+c^2)}};$$

$$\therefore x''=\frac{a}{12(a^2+c^2)}\sec\theta,\quad y''=\frac{a}{12(a^2+c^2)}\operatorname{cosec}\theta,\quad z''=0.$$

The equations of the principal normal are therefore

$$\frac{x-4a\cos^3\theta}{\sin\theta}=\frac{y-4a\sin^3\theta}{\cos\theta}=\frac{z-3c\cos 2\theta}{0}.$$

The equation of the osculating plane is

$$\begin{vmatrix} x-4a\cos^3\theta, & y-4a\sin^3\theta, & z-3c\cos 2\theta \\ -a\cos\theta, & a\sin\theta, & -c \\ \sin\theta, & \cos\theta, & 0 \end{vmatrix}=0.$$

Ex. 3. *To find to the third order the co-ordinates of any point of a curve in terms of the arc, when the axes of co-ordinates are the tangent, the principal normal, and the binormal at the point from which the arc is measured.*

Let OX, OY, OZ be the tangent, principal normal, and binormal at the point O of a curve. Let x, y, z be the co-ordinates of a point at a distance s from O, and let $\frac{1}{\rho}$ and $\frac{1}{\sigma}$ be the curvature and torsion of the curve at O.

Then, at the origin, $\quad x'=1,\quad y'=0,\quad z'=0;$

also $\quad \rho x''=0,\ \rho y''=1,\ z''=0.$

We have, at any point of the curve,

$$x'x''+y'y''+z'z''=0.$$

Differentiating, we have

$$\frac{1}{\rho^2}+x'x'''+y'y'''+z'z'''=0 \qquad \text{(i)}.$$

Also, by differentiating

$$\frac{1}{\rho^2}=x''^2+y''^2+z''^2,$$

we have at any point

$$-\frac{1}{\rho^3}\frac{d\rho}{ds}=x''x'''+y''y'''+z''z''' \qquad \text{(ii)}.$$

Also we know that

$$\frac{1}{\rho^2\sigma}=\begin{vmatrix} x', & y', & z' \\ x'', & y'', & z'' \\ x''', & y''', & z''' \end{vmatrix} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}.$$

From (i), (ii), (iii) we see that at the origin

$$x'''=-\frac{1}{\rho^2},\ y'''=-\frac{1}{\rho^2}\frac{d\rho}{ds},\ z'''=\frac{1}{\rho\sigma}.$$

Hence, by Maclaurin's Theorem, we have to the third order

$$x=s-\frac{s^3}{6\rho^2},\ y=\frac{s^2}{2\rho}-\frac{s^3}{6\rho^2}\frac{d\rho}{ds},\ z=\frac{s^3}{6\rho\sigma}.$$

Examples on Chapter XI.

1. Find the equation of the surface generated by the principal normals of a helix.

2. Find the osculating plane at any point of the curve

$$x=a\cos\theta+b\sin\theta,\ y=a\sin\theta+b\cos\theta,\ z=c\sin 2\theta,$$

and shew that it is always inclined at the same angle to the axis of z.

3. Find the equations of the principal normal at any point of the curve

$$x^2+y^2=a^2,\ \ az=x^2-y^2.$$

4. A point moves on an ellipsoid so that its direction of motion always passes through the perpendicular from the centre of the ellipsoid on the tangent plane at any point; shew that the curve traced out by the point is given by the intersection of the ellipsoid with the surface

$$x^{m-n}\,y^{n-l}\,z^{l-m}=\text{constant},$$

l, m, n being inversely proportional to the squares of the semi-axes of the ellipsoid.

5. A curve is traced on a right cone so as to cut all the generating lines at the same angle; shew that its projection on the plane of the base is an equiangular spiral.

6. Shew that any curve has an infinite number of evolutes which lie on its polar developable. Shew also that the locus of the centre of principal curvature is not an evolute.

7. If a circular helix be drawn passing through four consecutive points of a curve in space, prove that when the four points ultimately coincide the radius of the helix equals $\frac{\rho\sigma^2}{\rho^2+\sigma^2}$, and its slope is $\tan^{-1}\frac{\rho}{\sigma}$.

8. Shew that if the osculating plane at every point of a curve pass through a fixed point, the curve will be plane. Hence prove that the curves of intersection of the surfaces whose equations are $x^2+y^2+z^2=a^2$, and $x^4+y^4+z^4=\frac{a^4}{2}$ are circles of radius a.

9. Prove that the helix is the only curve whose radius of circular curvature and radius of torsion are both constant.

10. A curve is drawn on the cylinder whose equation is

$$b^2x^2+a^2y^2-a^2b^2=0,$$

cutting all the generators at an angle α; shew that its radius of curvature at any point is $\rho\operatorname{cosec}^2\alpha$, where ρ is the radius of curvature of the principal elliptic section through the point.

11. If a curve in space is defined by the equations

$$x=2a\cos t,\ \ y=2a\sin t,\ \ z=bt^2,$$

prove that the radius of circular curvature is equal to

$$\frac{2}{a}\sqrt{\left\{\frac{(a^2+b^2t^2)^3}{a^2+b^2+b^2t^2}\right\}}.$$

12. In any curve if R be the radius of spherical curvature, ρ the radius of absolute curvature and $\frac{1}{\sigma}$ the tortuosity at any point (x, y, z), then

$$\rho^4\left\{\left(\frac{d^3x}{ds^3}\right)^2+\left(\frac{d^3y}{ds^3}\right)^2+\left(\frac{d^3z}{ds^3}\right)^2\right\}=1+\frac{R^2}{\sigma^2}.$$

13. If the tangent and the normal to the osculating plane at any point of a curve make angles α, β with any fixed line in space, shew that $\frac{\sin\alpha}{\sin\beta}\cdot\frac{d\alpha}{d\beta}=\frac{\sigma}{\rho}$, where $\frac{1}{\rho}$, $\frac{1}{\sigma}$ are the curvature and tortuosity respectively.

14. Find the curvature and torsion at any point of the curve in question 5.

15. Prove that the origin is the centre of absolute curvature of the curve $ax^2 + by^2 + cz^2 = 1$, $rx^2 + ry^2 + rz^2 = 1$ at all points, whose co-ordinates satisfy the equation

$$\frac{a-r}{b-c}x^4 + \frac{b-r}{c-a}y^4 + \frac{c-r}{a-b}z^4 = 0.$$

16. A curve is drawn on a right circular cone always inclined at the same angle α to the axis; prove that $\sigma = \rho \tan \alpha$.

17. If ρ, σ be the radii of curvature and torsion at any point of a curve in space; ρ', σ' similar quantities at the corresponding point of the locus of the centre of spherical curvature, then

$$\rho\rho' = \sigma\sigma'.$$

18. Every portion of a curve is equal and similar to the corresponding portion of the edge of regression of the polar surface; prove that the tangent to it makes an angle of 45^0 with a fixed plane, and that its projection on that plane is the evolute of a circle.

19. Shew that if along the tangent to any curve a point be taken at a constant distance c from the point of contact of the tangent to the given curve, and if ρ_1 be the radius of curvature in the osculating plane of the curve traced out by the point, then

$$\frac{(c^2+\rho^2)^3}{\rho_1^{\,2}} = \frac{c^2\rho^2\,(c^2+\rho^2)}{\sigma^2} + \left(c^2+\rho^2 - c\rho\,\frac{d\rho}{ds}\right)^2,$$

where ρ and σ are the radii of curvature and torsion of the given curve.

20. A circle of radius a is traced on a piece of paper, which is then folded so as to become a cylinder of radius b; shew that, if ρ be the radius of curvature at any point of the curve which the circle now becomes, then $\frac{1}{\rho^2} = \frac{1}{a^2} + \frac{1}{b^2}\cos^4\frac{s}{a}$, where s is the distance, measured along the arc, of the point from a certain fixed point of the curve.

CHAPTER XII.

Curvature of Surfaces.

241. We have already seen, in Art. 209, that the section of any surface, by a plane parallel to and indefinitely near the tangent plane at any point O on the surface, is a conic, which is called the *Indicatrix*, and whose centre is on the normal at O.

242. Let any section of the surface, drawn through the normal OV, cut the indicatrix in the diameter QVQ', and let ρ be the radius of curvature at O of the section. Then we have, in the limit, $2\rho \,.\, OV = QV^2$. Hence, for different normal sections through O, the radius of curvature varies as the square of the diameter of the indicatrix through which the section passes.

243. Since the sum of the squares of the reciprocals of any two perpendicular semi-diameters of a conic is constant, it follows from the last article that the sum of the reciprocals of the radii of curvature of any two perpendicular normal sections through a given point of a surface is constant.

244. Since the semi-diameter of a conic has a maximum and a minimum value, it follows from Art. 242 that the radius of curvature of a normal section through any point of a surface has a maximum and a minimum value, the corresponding sections being those which pass through the axes of the indicatrix.

The maximum and minimum radii of curvature are called the *principal radii* of curvature, and the corresponding normal sections are called the *principal sections.*

The locus of the centres of principal curvature at all points of a given surface is called its *surface of centres.*

245. If the axes of x and y be taken in the direction of the axes of the indicatrix the equation of the surface will be, when the terms of the third and higher orders are neglected,

$$2z = ax^2 + by^2.$$

Let ρ_1, ρ_2 be the principal radii of curvature, that is the radii of curvature of the sections made by the planes $y = 0$, $x = 0$ respectively; then it is clear that $\rho_1 = \frac{1}{a}$, and $\rho_2 = \frac{1}{b}$. Hence the equation of the surface will be

$$2z = \frac{x^2}{\rho_1} + \frac{y^2}{\rho_2}.$$

The semi-diameter of the indicatrix which makes an angle θ with the axis of x is given by

$$\frac{2z}{r^2} = \frac{\cos^2\theta}{\rho_1} + \frac{\sin^2\theta}{\rho_2}.$$

If ρ be the radius of curvature of the corresponding section, we have $r^2 = 2\rho z$.

Hence $$\frac{1}{\rho} = \frac{\cos^2\theta}{\rho_1} + \frac{\sin^2\theta}{\rho_2}.$$

The results of Articles 243, 244 and 245 are due to Euler.

246. When the indicatrix at any point of a surface is an ellipse, the sign of the radius of curvature is the same for all sections; this shews that the concavity of all sections is turned in the same direction, so that the surface, in the neighbourhood of the point, is entirely on one side of the tangent plane. The surface in this case is said to be *Synclastic* at the point.

When the indicatrix is an hyperbola, the sign of the radius of curvature is sometimes positive and sometimes negative, shewing that the concavity of some sections is turned in opposite directions to that of others. The surface in this case is said to be *Anticlastic* at the point.

The radius of curvature of a section which passes through an asymptote of the indicatrix is infinite; hence the asymptotes divide the sections whose concavity is turned one way from those whose concavity is turned the other way.

In the figure of Art. 71, the concavities of the sections by the planes $x=0$ and $y=0$ are turned in opposite directions; and the normal sections through the two generating lines at O are the sections of zero curvature.

When the indicatrix is a parabola, that is to say is two parallel straight lines, which become ultimately coincident, one of the principal radii of curvature is infinite; and, if ρ_1 be the finite radius of principal curvature, the curvature of any other normal section is given by the formula $\dfrac{1}{\rho} = \dfrac{\cos^2\theta}{\rho_1}$.

247. *To find the radius of curvature of any oblique section of a surface.*

Let any oblique section through the point O of a surface cut the indicatrix in the line RKR', and let the normal section through the same tangent line cut the indicatrix in the line QVQ' parallel to RKR'. Let K, V be the middle points of RR', QQ' respectively, and let ρ, ρ_0 be the radii of curvature of the sections ROR', QOQ' respectively.

Then we have, in the limit,

$$2\rho \,.\, OK = RK^2,$$

and

$$2\rho_0 \,.\, OV = QV^2.$$

But OV, and therefore VK, is small compared with QV; hence RR' and QQ' are ultimately equal. Also

$$OV = OK \cos\theta,$$

where θ is the angle between the planes ROR' and QOQ'.

Hence we have ultimately,

$$\frac{\rho}{\rho_0} = \frac{OV}{OK} = \cos\theta,$$

or

$$\rho = \rho_0 \cos\theta.$$

This is called Meunier's Theorem.

248. From Meunier's Theorem, and the theorem of Art. 245, it follows that if two surfaces touch one another, and have the same radii of principal curvature at the point of contact, then all sections through that point have the same curvature.

249. The following proof of Meunier's Theorem is due to Dr Besant.

Let OT be any tangent line at the point O of a surface, and let P be a point contiguous to O on the normal section through OT, and Q a point contiguous to O on an oblique section through OT. Then a sphere can be described to touch OT at O, and to pass through P and Q; and the sections of this sphere by the planes TOQ, TOP are ultimately the circles of curvature at O of the sections of the surface by those planes. Hence, as Meunier's Theorem is obviously true for a sphere, it is true for the surface.

Ex. 1. Find the principal radii of curvature at the origin of the surface $2z = 6x^2 - 5xy - 6y^2$. *Ans.* $\frac{2}{13}$, $-\frac{2}{13}$.

Ex. 2. Find the radius of principal curvature at any point of the curve of intersection of two surfaces.

Let ρ be the required radius of curvature at any point P. Let the surfaces intersect at an angle α, and let θ, $\alpha - \theta$ be the angles between the principal normal of the curve of intersection, and the normals to the two surfaces. Let ρ_1, ρ_2 be the radii of curvature of normal sections of the two surfaces through the tangent line at P. Then, by Meunier's Theorem,

$$\rho = \rho_1 \cos\theta, \text{ and } \rho = \rho_2 \cos(\alpha - \theta).$$

Hence, eliminating θ, we have

$$\frac{\sin^2\alpha}{\rho^2} = \frac{1}{\rho_1^{\,2}} + \frac{1}{\rho_2^{\,2}} - \frac{2\cos\alpha}{\rho_1\rho_2}.$$

250. Def. A *line of curvature* on any surface is a curve such that the tangent line to it at any point is a tangent line to one of the principal sections of the surface at that point.

251. *The normals to any surface at consecutive points of one of its lines of curvature intersect.*

Let P be an extremity of an axis of the indicatrix which corresponds to the point O of a surface, then O, P are consecutive points on a line of curvature.

Let V be the centre of the indicatrix, then OV will be the normal to the surface at O.

The tangent line at P to the indicatrix is perpendicular to the normal to the surface at P; it is also perpendicular to OV; and, since P is an extremity of an axis of the indicatrix, the tangent line is perpendicular to PV. Hence OV, PV, and the normal at P are in a plane, and therefore the normals at O and P will intersect.

Conversely, if the normals at P and O intersect, the tangent line at P to the indicatrix will be perpendicular to the plane which contains the normals at O and P; therefore the tangent line will be perpendicular to PV, and hence PV is an axis of the indicatrix.

252. *To find the differential equations of the lines of curvature on any surface.*

Let $F(x, y, z) = 0$ be the equation of the surface. Then the equations of the normal at any point (x, y, z) are

$$\frac{\xi - x}{\frac{dF}{dx}} = \frac{\eta - y}{\frac{dF}{dy}} = \frac{\zeta - z}{\frac{dF}{dz}}.$$

The normal at the consecutive point

$(x + dx,\ y + dy,\ z + dz)$ is

$$\frac{\xi - x - dx}{\frac{dF}{dx} + d\left(\frac{dF}{dx}\right)} = \frac{\eta - y - dy}{\frac{dF}{dy} + d\left(\frac{dF}{dy}\right)} = \frac{\zeta - z - dz}{\frac{dF}{dz} + d\left(\frac{dF}{dz}\right)}.$$

The condition of intersection of the two normals gives the equation

$$\begin{vmatrix} dx, & dy, & dz \\ \frac{dF}{dx}, & \frac{dF}{dy}, & \frac{dF}{dz} \\ d\left(\frac{dF}{dx}\right), & d\left(\frac{dF}{dy}\right), & d\left(\frac{dF}{dz}\right) \end{vmatrix} = 0 \ldots \text{(i)}.$$

Since $(x + dx,\ y + dy,\ z + dz)$ is on the surface, we have also

$$\frac{dF}{dx}dx + \frac{dF}{dy}dy + \frac{dF}{dz}dz = 0 \ldots \text{(ii)}.$$

The equations (i) and (ii) are the required differential equations.

253. *To find the principal radii of curvature, and the lines of curvature, on a surface of revolution.*

It is clear that the normals to the surface at all points on a meridian lie in the plane through the axis and that meridian; hence normals at consecutive points on a meridian intersect, so that any meridian is a line of curvature. It is also clear that the normals to the surface at all points of any circle whose plane is perpendicular to the axis of the surface, meet the axis in the same point, and therefore any such circle is a line of curvature. Hence the lines of curvature are the meridians, and the circular sections which are perpendicular to the axis.

It is easy to see that one of the principal radii at any point P is the radius of curvature of the generating curve at P; and that the other principal radius is the length of the normal intercepted between P and the axis.

254. The tangent plane to a developable touches the surface at all points of a generating line. The normals to the surface at all points of a generating line are therefore parallel; hence normals at consecutive points intersect, so that one set of the lines of curvature of a developable are the

generating lines, the corresponding radii of curvature being infinite.

The other lines of curvature are curves which cut all the generating lines perpendicularly; and hence, if the surface be developed into a plane, the lines of curvature will become involutes of the curve into which the edge of regression developes.

In the particular case of the developable being a cone, the lines of curvature will cut the generating lines at a constant distance from the vertex, and hence they are the curves of intersection of the surface and spheres with the vertex for centre.

Ex. 1. Find the surface of revolution which is such that the indicatrix at any point is a rectangular hyperbola.

The principal radii of curvature must be equal and opposite at any point. Hence the radius of curvature at any point of the generating curve must be equal and opposite to the normal: this is a known property of a catenary. Hence the surface is that formed by the revolution of a catenary about its axis.

Ex. 2. Shew from the general differential equations of lines of curvature, that one system of lines of curvature on a cone are the generating lines, and the other system are the curves of intersection of the surface and concentric spheres.

The equations are

$$\begin{vmatrix} dx\,, & dy\,, & dz \\ \dfrac{dF}{dx}\,, & \dfrac{dF}{dy}\,, & \dfrac{dF}{dz} \\ d\left(\dfrac{dF}{dx}\right), & d\left(\dfrac{dF}{dy}\right), & d\left(\dfrac{dF}{dz}\right) \end{vmatrix} = 0 \quad\text{............(i)},$$

and

$$\frac{dF}{dx}dx + \frac{dF}{dy}dy + \frac{dF}{dz}dz = 0 \quad\text{......................(ii)}.$$

Since the surface is a cone whose vertex is at the origin, we have

$$x\frac{dF}{dx} + y\frac{dF}{dy} + z\frac{dF}{dz} = 0 \quad\text{........................(iii)},$$

therefore from (ii)

$$xd\left(\frac{dF}{dx}\right) + yd\left(\frac{dF}{dy}\right) + zd\left(\frac{dF}{dz}\right) = 0 \quad\text{.............. (iv)}.$$

Multiply the terms of the columns in (i) by x, y, z respectively, and add; then on account of (iii) and (iv), (i) will become

$$\begin{vmatrix} dx\ , & dy\ , & x\,dx+y\,dy+z\,dz \\ \dfrac{dF}{dx}\ , & \dfrac{dF}{dy} & 0 \\ d\left(\dfrac{dF}{dx}\right), & d\left(\dfrac{dF}{dy}\right), & 0 \end{vmatrix}=0.$$

Hence either $$x\,dx+y\,dy+z\,dz=0 \qquad \text{(v)},$$

or $$\frac{d\left(\dfrac{dF}{dx}\right)}{\dfrac{dF}{dx}}=\frac{d\left(\dfrac{dF}{dy}\right)}{\dfrac{dF}{dy}}=\frac{d\left(\dfrac{dF}{dz}\right)}{\dfrac{dF}{dz}} \qquad \text{(vi)}.$$

From (v) we have $\qquad x^2+y^2+z^2=\text{constant},$

shewing that one series of the lines of curvature are the curves of intersection of the surface and concentric spheres.

From (vi) we have

$$\frac{\dfrac{dF}{dx}}{l}=\frac{\dfrac{dF}{dy}}{m}=\frac{\dfrac{dF}{dz}}{n},$$

where l, m, n are constants. Hence, from (iii), we have

$$lx+my+nz=0,$$

which shews that the other series of lines of curvature are the generating lines.

Ex. 3. If two surfaces cut one another at a constant angle, and the curve of intersection be a line of curvature on one of the surfaces, it will be a line of curvature on the other.

Let P, Q be any two consecutive points on the curve of intersection, and let Oab be the line of intersection of the normal planes of the curve at P, Q, where O is in the osculating plane of the arc PQ. If the curve of intersection be a line of curvature on one of the surfaces, the normals to that surface at P, Q must intersect, they will therefore meet the line Oab in the same point, a suppose.

Let the normals to the other surface at P, Q meet Oab in c, c' respectively.

The triangles OPa, OQa are equal in all respects, for $PO=QO$, $Pa=Qa$, and Oa is common. And, since the surfaces intersect at a constant angle, the angles aPc and aQc' are equal. Therefore the angle OPc, OQc' are equal. But the angles POc, QOc' are equal, and $PO=QO$. Therefore $Oc=Oc'$. This proves the proposition.

Ex. 4. If the line of intersection of two surfaces be a line of curvature on both, the two surfaces cut at a constant angle.

For let P, Q be any two consecutive points on the curve of intersection; let the normals to one surface at P, Q meet in a, and the normals to the other surface meet in b. Then, we have $Pa=Qa$, $Pb=Qb$, and ab common to the two triangles aPb, aQb. Hence the angles aPb and aQb are equal.

Ex. 5. If a line of curvature be a plane curve its plane will cut the surface at a constant angle.

Any line is a line of curvature on a plane (or on a sphere). The theorem therefore is a particular case of Ex. 4.

255. *If three series of surfaces intersect at right angles at all their common points, the curve of intersection of any two is a line of curvature on each.* (Dupin's Theorem.)

Take for origin a point of intersection of three of the surfaces, one of each series, and let the three perpendicular tangent planes be taken for co-ordinate planes. The equations of the three surfaces will then be

$$2x + ay^2 + bz^2 + 2hyz + \ldots\ldots\ldots = 0 \ldots\ldots\ldots \text{(i)},$$
$$2y + a'z^2 + b'x^2 + 2h'zx + \ldots\ldots\ldots = 0 \ldots\ldots\ldots \text{(ii)},$$
$$2z + a''x^2 + b''y^2 + 2h''xy + \ldots\ldots\ldots = 0 \ldots\ldots\ldots \text{(iii)}.$$

At a consecutive point common to (i) and (ii) we have $x = 0$, $y = 0$, $z = z'$, where z' is very small; and the tangent planes to (i) and (ii) at $(0, 0, z')$ are ultimately

$$x + bzz' + hyz' = 0,$$
$$y + a'zz' + h'xz' = 0.$$

The condition that these may be at right angles gives

$$h'z' + hz' + a'bz'^2 = 0,$$

or, ultimately, $h + h' = 0$. We have similarly, since the other surfaces cut at right angles, $h' + h'' = 0$, and $h'' + h = 0$. Hence $h = h' = h'' = 0$, and therefore the axes are tangents to the lines of curvature on each surface. This being true at all points of intersection of three surfaces, it follows that all curves of intersection of two surfaces of different systems are lines of curvature on each.

We have proved in Art. 164 that confocal conicoids cut one another at right angles at all their common points. Hence, one system of the lines of curvature of an ellipsoid are its curves of intersection with confocal hyperboloids of one sheet, and the other system of lines of curvature are the curves of intersection with confocal hyperboloids of two sheets.

256. *To find the principal radii of curvature at any point of a surface.*

Let ξ, η, ζ be the co-ordinates of the point of intersection of the normals at two consecutive points (x, y, z) and $(x+dx, y+dy, z+dz)$ of a surface, and let ρ be the radius of curvature at (x, y, z) of the normal section through those points. Then [Art. 251] ρ is one of the principal radii of curvature, and we have

$$\frac{\xi - x}{\frac{dF}{dx}} = \frac{\eta - y}{\frac{dF}{dy}} = \frac{\zeta - z}{\frac{dF}{dz}} = \frac{\rho}{\sqrt{\left\{\left(\frac{dF}{dx}\right)^2 + \left(\frac{dF}{dy}\right)^2 + \left(\frac{dF}{dz}\right)^2\right\}}} = \frac{\rho}{\kappa};$$

$$\therefore\ \xi = x + \frac{\rho}{\kappa}\frac{dF}{dx},\ \eta = y + \frac{\rho}{\kappa}\frac{dF}{dy},\ \zeta = z + \frac{\rho}{\kappa}\frac{dF}{dz}.$$

And, since (ξ, η, ζ) is also on the normal at $(x+dx, y+dy, z+dz)$, we have by differentiating the preceding equations, considering ξ, η, ζ, ρ as constant,

$$0 = dx + \frac{\rho}{\kappa} d\left(\frac{dF}{dx}\right) - \frac{\rho d\kappa}{\kappa^2}\frac{dF}{dx},$$

and two similar equations.

Since

$$d\left(\frac{dF}{dx}\right) = \frac{d^2F}{dx^2}dx + \frac{d^2F}{dxdy}dy + \frac{d^2F}{dxdz}dz,$$

and similarly for $d\left(\frac{dF}{dy}\right)$ and $d\left(\frac{dF}{dz}\right)$, the equations may be written

$$0 = \left(\frac{\kappa}{\rho} + \frac{d^2F}{dx^2}\right)dx + \frac{d^2F}{dxdy}dy + \frac{d^2F}{dxdz}dz - \frac{d\kappa}{\kappa}\frac{dF}{dx},$$

$$0 = \frac{d^2F}{dxdy}dx + \left(\frac{\kappa}{\rho} + \frac{d^2F}{dy^2}\right)dy + \frac{d^2F}{dydz}dz - \frac{d\kappa}{\kappa}\frac{dF}{dy},$$

$$0 = \frac{d^2F}{dxdz}dx + \frac{d^2F}{dydz}dy + \left(\frac{\kappa}{\rho} + \frac{d^2F}{dz^2}\right)dz - \frac{d\kappa}{\kappa}\frac{dF}{dz}.$$

We have also

$$0 = \frac{dF}{dx}dx + \frac{dF}{dy}dy + \frac{dF}{dz}dz.$$

Eliminating dx, dy, dz, $d\kappa$ we have for the determination of the principal radii the equation

$$\begin{vmatrix} \frac{\kappa}{\rho}+\frac{d^2F}{dx^2}, & \frac{d^2F}{dxdy}, & \frac{d^2F}{dxdz}, & \frac{dF}{dx} \\ \frac{d^2F}{dxdy}, & \frac{\kappa}{\rho}+\frac{d^2F}{dy^2}, & \frac{d^2F}{dydz}, & \frac{dF}{dy} \\ \frac{d^2F}{dxdz}, & \frac{d^2F}{dydz}, & \frac{\kappa}{\rho}+\frac{d^2F}{dz^2}, & \frac{dF}{dz} \\ \frac{dF}{dx}, & \frac{dF}{dy}, & \frac{dF}{dz}, & 0 \end{vmatrix} = 0.$$

257. *To find the umbilics of any surface.*

At an umbilic the indicatrix is a circle.

Let the equation of the surface be $F(x, y, z) = 0$, and let (x', y', z') be any point on it. The equation of the surface referred to parallel axes through (x', y', z') will be

$$x\frac{dF}{dx'}+y\frac{dF}{dy'}+z\frac{dF}{dz'}+\tfrac{1}{2}\left(x\frac{d}{dx'}+y\frac{d}{dy'}+z\frac{d}{dz'}\right)^2 F+\ldots=0.$$

Hence the indicatrix is similar to the section of the conicoid

$$\frac{d^2F}{dx'^2}x^2+\frac{d^2F}{dy'^2}y^2+\frac{d^2F}{dz'^2}z^2+2\frac{d^2F}{dy'dz'}yz$$
$$+2\frac{d^2F}{dz'dy'}zx+2\frac{d^2F}{dx'dy'}xy+1=0 \quad \ldots\text{(i)},$$

by the plane

$$x\frac{dF}{dx'}+y\frac{dF}{dy'}+z\frac{dF}{dz'}=0\ldots\ldots\ldots\ldots\ldots\text{(ii)},$$

and we have already found [Art. 125, Ex. 5] the conditions that a given section of a conicoid may be circular.

From the result of Art. 256 it is clear that the two values of $\frac{\rho}{\kappa}$ are the squares of the axes of the section of (i) by (ii).

258. *To find the radii of principal curvature, and the lines of curvature, of the surface whose equation is $z = f(x, y)$.*

Let (ξ, η, ζ) be one of the centres of principal curvature at the point (x, y, z), and let ρ be the corresponding radius of curvature. Then, the equations of the normal at (x, y, z) will be

$$\frac{\xi - x}{p} = \frac{\eta - y}{q} = \frac{\zeta - z}{-1} = \frac{\rho}{\sqrt{(1 + p^2 + q^2)}},$$

therefore $\qquad \xi - x = -p(\zeta - z),$

and $\qquad \eta - y = -q(\zeta - z).$

Since the normal at $(x + dx, y + dy, z + dz)$ also passes through (ξ, η, ζ) we have, by differentiating the preceding equations,

$$-dx = -dp(\zeta - z) + pdz,$$

and $\qquad -dy = -dq(\zeta - z) + qdz;$

that is $\qquad -dx = p(pdx + qdy) - (\zeta - z)(rdx + sdy) \ldots \text{(i)}$

and $\qquad -dy = q(pdx + qdy) - (\zeta - z)(sdx + tdy) \ldots \text{(ii)}.$

Eliminating $\zeta - z$ from (i) and (ii) we have

$$\frac{(1 + p^2)dx + pqdy}{rdx + sdy} = \frac{pqdx + (1 + q^2)dy}{sdx + tdy};$$

therefore $\quad (1 + p^2)s - pqr + \{(1 + p^2)t - (1 + q^2)r\}\dfrac{dy}{dx}$

$$+ \{pqt - s(1 + q^2)\}\left(\frac{dy}{dx}\right)^2 = 0 \ldots \text{(iii)},$$

which is the differential equation of the projection of the lines of curvature on the plane $z = 0$.

Again, from (i) and (ii) we have, putting κ for

$$\sqrt{1 + p^2 + q^2},$$

$$\left(1 + p^2 + \frac{r\rho}{\kappa}\right)dx + \left(pq + \frac{s\rho}{\kappa}\right)dy = 0,$$

and $\qquad \left(pq + \dfrac{s\rho}{\kappa}\right)dx + \left(1 + q^2 + \dfrac{t\rho}{\kappa}\right)dy = 0.$

Hence $$\left(1+p^2+\frac{r\rho}{\kappa}\right)\left(1+q^2+\frac{t\rho}{\kappa}\right)-\left(pq+\frac{s\rho}{\kappa}\right)^2=0,$$
or
$$(rt-s^2)\,\rho^2+\kappa\,\{t\,(1+p^2)+r\,(1+q^2)-2pqs\}\,\rho+\kappa^2=0\ldots\text{(iv)},$$
which is an equation giving the principal radii of curvature.

259. At an umbilicus the directions of principal curvature are indeterminate; hence the conditions for an umbilicus are, from equation (iii) of the last Article,
$$\frac{1+p^2}{r}=\frac{1+q^2}{t}=\frac{pq}{s}.$$

260. Def. The *whole curvature* of any portion of a surface, bounded by a closed curve, is equal to the area cut off from a sphere of unit radius by radii which are parallel to the normals to the surface at all points of the curve.

The *average curvature* of any portion of a surface is the ratio of the whole curvature to the area of that portion.

The *measure of curvature* at any point is the average curvature of a very small portion which includes the point.

These definitions, which are analogous to the definitions in plane curves, are due to Gauss.

The curve traced out on the unit sphere as above is called the *horograph* of the given portion of the surface.

261. *To shew that the measure of curvature at any point of a surface is the reciprocal of the product of the principal radii of curvature of the surface at that point.*

Consider a small portion $PQRS$ of the surface bounded by lines of curvature; then $PQRS$ is ultimately a rectangle whose area is $PQ\,.\,PS$.

Let lines parallel to the normals at P, Q, R, S, drawn through the centre of a sphere of unit radius, meet the sphere in p, q, r, s. Then, since the principal planes at any point of a surface are at right angles, the angles p, q, r, s are right angles, and therefore $pqrs$ is ultimately a rectangle whose area is $pq\,.\,ps$. But the angle between the normals at P and Q

is ultimately $\frac{PQ}{\rho_1}$, and the angle between the normals at P and S is ultimately $\frac{PS}{\rho_2}$, where ρ_1, ρ_2 are the principal radii of curvature at P. Hence $pq = \frac{PQ}{\rho_1}$, and $ps = \frac{PS}{\rho_2}$, so that the area of $pqrs$ is ultimately $\frac{PQ \, . \, PS}{\rho_1 \rho_2}$. Hence the measure of curvature at P, which by definition is the limiting value of $\frac{\text{area } pqrs}{\text{area } PQRS}$, is $\frac{1}{\rho_1 \rho_2}$.

Geodesic Lines.

262. Def. A *geodesic line* on a surface is such that any small element AB is the shortest line which can be drawn on the surface from A to B.

The length of the line joining any two indefinitely near points will clearly be least when the curvature is least. But by Meunier's theorem, the curvature of a surface through a given tangent line is least when the section is a normal section. Hence at any point of a geodesic line on a surface the plane of the curve contains the normal to the surface, so that the principal normal of the curve coincides with the normal to the surface. We therefore have at any point of a geodesic line on a surface

$$\frac{\dfrac{d^2x}{ds^2}}{\dfrac{dF}{dx}} = \frac{\dfrac{d^2y}{ds^2}}{\dfrac{dF}{dy}} = \frac{\dfrac{d^2z}{ds^2}}{\dfrac{dF}{dz}}.$$

Curvature of Conicoids.

263. Since all parallel sections of a conicoid are similar, it follows that the indicatrix at any point P of a conicoid is similar to the central section which is parallel to the tangent plane at P. Hence the tangents to the lines of curvature at any point P are parallel to the axes of that central section.

Now, by Art. 167, the lines which are parallel to the axes of the central section are the tangent lines at P to the curves of intersection of the conicoid with the confocals which go through P. Hence, as we have already proved in Art. 255, the *lines of curvature of a conicoid are the curves of intersection with confocal conicoids.*

264. We can shew that the lines of curvature on a conicoid are its curves of intersection with confocals in the following manner.

At points common to

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=1 \quad\text{.........................(i)},$$

and

$$\frac{x^2}{a+\lambda}+\frac{y^2}{b+\lambda}+\frac{z^2}{c+\lambda}=1 \quad\text{..............(ii)},$$

we have, by subtraction,

$$\frac{x^2}{a(a+\lambda)}+\frac{y^2}{b(b+\lambda)}+\frac{z^2}{c(c+\lambda)}=0 \quad\text{.........(iii)}.$$

Differentiating (ii) and (iii) we have

$$\frac{xdx}{a+\lambda}+\frac{ydy}{b+\lambda}+\frac{zdz}{c+\lambda}=0 \quad\text{....................(iv)},$$

and

$$\frac{xdx}{a(a+\lambda)}+\frac{ydy}{b(b+\lambda)}+\frac{zdz}{c(c+\lambda)}=0 \quad\text{............(v)}.$$

The elimination of $a+\lambda$, $b+\lambda$, $c+\lambda$ from (iii), (iv), (v) gives

$$\begin{vmatrix} \dfrac{x}{a}, & \dfrac{y}{b}, & \dfrac{z}{c} \\ dx, & dy, & dz \\ \dfrac{dx}{a}, & \dfrac{dy}{b}, & \dfrac{dz}{c} \end{vmatrix}=0 \quad\text{..................(vi)},$$

which is the differential equation of the curve of intersection of (i) and any one of its confocals; and it is easy to see, by comparing with (i), Art. 252, that (vi) is the differential equation of a line of curvature.

265. The radius of curvature of any normal section of a central conicoid may be found as follows.

The radius of curvature of any central section of a conicoid through a point P is, by a well-known formula, equal to $\frac{d^2}{p}$, where d is the semi-diameter parallel to the tangent at P, and p is the perpendicular from the centre on the tangent at P. Hence, by Meunier's Theorem, the radius of curvature of any normal section of a conicoid through the point P is equal to $\frac{d^2}{p_0}$, where p_0 is the perpendicular from the centre on the tangent plane at P, and d is the semi-diameter parallel to the tangent line at P; for the cosine of the angle between the normal section and the central section is $\frac{p_0}{p}$.

266. *At any point of a line of curvature of a central conicoid, the rectangle contained by the diameter parallel to the tangent at that point and the perpendicular from the centre on the tangent plane at the point is constant.*

Let p be the perpendicular from the centre on the tangent plane at any point P of a given line of curvature, and let α, β be the semi-axes of the central section parallel to the tangent plane at P. Then, one of the axes, α suppose, is parallel to the tangent at P to the line of curvature, and the other axis is of constant length for all points on the line of curvature [Art. 167, Cor.]. Hence, since $p\alpha\beta$ is constant, it follows that $p\alpha$ is constant throughout the line of curvature.

267. *At any point of a geodesic on a central conicoid, the rectangle contained by the diameter parallel to the tangent at that point and the perpendicular from the centre on the tangent plane at the point is constant.*

The differential equations of a geodesic on the conicoid $ax^2 + by^2 + cz^2 = 1$ are

$$\frac{\frac{d^2x}{ds^2}}{ax} = \frac{\frac{d^2y}{ds^2}}{by} = \frac{\frac{d^2z}{ds^2}}{cz},$$

or
$$\frac{x''}{ax}=\frac{y''}{by}=\frac{z''}{cz}=\lambda \qquad\text{(i).}$$

We have to prove that pr is constant, where
$$\frac{1}{r^2}=ax'^2+by'^2+cz'^2 \qquad\text{(ii),}$$
and
$$\frac{1}{p^2}=a^2x^2+b^2y^2+c^2z^2 \qquad\text{(iii).}$$

Differentiating $ax^2+by^2+cz^2=1$ twice with respect to s, we have
$$ax'^2+by'^2+cz'^2+axx''+byy''+czz''=0 \qquad\text{(iv).}$$

From (i) we have
$$\lambda=\frac{axx''+byy''+czz''}{a^2x^2+b^2y^2+c^2z^2}=-\frac{p^2}{r^2},\ \text{from (iii) and (iv).}$$

$$\text{Also } \lambda=\frac{ax'x''+by'y''+cz'z''}{a^2xx'+b^2yy'+c^2zz'}=\frac{\dfrac{1}{r^3}\dfrac{dr}{ds}}{\dfrac{1}{p^3}\dfrac{dp}{ds}},\ \text{from (ii) and (iii).}$$

Hence
$$\frac{1}{r}\frac{dr}{ds}+\frac{1}{p}\frac{dp}{ds}=0,$$
and therefore pr is constant.

Ex. 1. The constant pr is the same for all geodesics which pass through an umbilic.

This follows from the fact that the central section parallel to the tangent plane at an umbilic is a circle, and therefore the semi-diameter parallel to the tangent to any geodesic through an umbilic is of constant length.

Ex. 2. The constant pr has the same value for all geodesics which touch the same line of curvature.

At the point of contact of the line of curvature and a geodesic which touches it, both p and r are the same for the line of curvature and for the geodesic.

Ex. 3. Two geodesics which touch the same line of curvature make equal angles with the lines of curvature through their point of intersection.

From Ex. 2, the semi-diameters parallel to the tangents to the two geodesics, at their point of intersection P, are equal to one another, and are therefore equally inclined to the axes of the central section which is parallel to the tangent plane at P. But the axes of the central section are parallel to the tangents to the lines of curvature through P: this proves the proposition.

Ex. 4. Two geodesics which pass through umbilics make equal angles with the lines of curvature through their point of intersection.

Ex. 5. Any geodesic through an umbilic will pass through the opposite umbilic.

Ex. 6. The locus of a point which moves so that the sum, or the difference, of its geodesic distances from two adjacent umbilics is constant, is a line of curvature.

Ex. 7. All geodesics which join two opposite umbilics are of constant length.

Ex. 8. The point of intersection of two geodesic tangents to a given line of curvature, which intersect at right angles, is on a sphere.

Let r_1, r_2 be the semi-diameters parallel to the tangents to the geodesics at P, their point of intersection. Then, since the geodesics cut at right angles,

$$\frac{1}{r_1^2}+\frac{1}{r_2^2}=\frac{1}{\alpha^2}+\frac{1}{\beta^2},$$

where α and β are the semi-axes of the central section parallel to the tangent plane at P. But, if p be the perpendicular on the tangent plane at P, then $pr_1=pr_2=$ constant, from Ex. 2. Hence, since $p\alpha\beta$ is constant, and also $\alpha^2+\beta^2+OP^2$, it follows that OP is constant.

Ex. 9. The point of intersection of two geodesic tangents, one to each of two given lines of curvature, which cut at right angles, is on a sphere.

Examples on Chapter XII.

1. A surface is formed by the revolution of a parabola about its directrix; shew that the principal curvatures at any point are in a constant ratio.

2. If ρ, ρ' be the principal radii of curvature of any point of an ellipsoid on the line of its intersection with a given concentric sphere, prove that the expression $\dfrac{(\rho\rho')^{\frac{1}{4}}}{\rho+\rho'}$ will be invariable.

3. If $u_1+u_2+u_3+\ldots\ldots u_n=0$ be the equation to a surface where u_r is a homogeneous function of x, y, z, of the rth degree, then $u_1+u_2+u_1(lx+my+nz)=0$ will be the general equation of surfaces of the second order having the same curvature at the origin.

4. The normal at each point of a principal section of an ellipsoid is intersected by the normal at a consecutive point not on the principal section; shew that the locus of the point of intersection is an ellipse having four (real or imaginary) contacts with the evolute of the principal section.

5. In the surface $y\cos\frac{z}{a}-x\sin\frac{z}{a}=0$,

the principal radii of curvature at (x, y, z) are $\pm\frac{x^2+y^2+a^2}{a}$.

6. Shew that the umbilici of the surface

$$\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}+\left(\frac{z}{c}\right)^{\frac{2}{3}}=1$$

lie on a sphere whose centre is the origin and whose radius is equal to $\frac{abc}{ab+bc+ca}$.

7. The centres of curvature of plane sections of a surface at any point lie on the surface

$$(x^2+y^2+z^2)\left(\frac{x^2}{\rho_1}+\frac{y^2}{\rho_2}\right)=z(x^2+y^2).$$

8. Prove that the line which separates the synclastic from the anticlastic parts of a surface is a line of curvature, and that along it the inflexional tangents coincide.

9. The projections of the lines of curvature of an ellipsoid on the cyclic planes, by lines parallel to the greatest axis of the surface, are confocal conics.

10. If one of the lines of curvature on a developable surface lies on a sphere all the other lines of curvature, other than the rectilineal ones, lie on concentric spheres.

11. A plane curve is wrapped upon a developable surface. If ρ is the radius of curvature of the plane curve at any point, ρ' the corresponding radius of circular curvature of the curve upon the surface, R the corresponding principal radius of curvature of the surface, and ϕ the angle at which the curve intersects the generator of the surface, $\frac{\sin^4\phi}{R^2}=\frac{1}{\rho'^2}-\frac{1}{\rho^2}$.

12. If one system of lines of curvature of a surface are circles, the surface is the envelope of a sphere whose centre moves on a given curve.

13. If a geodesic line is either a line of curvature or a plane curve it is both; but a plane line of curvature is not necessarily geodesic.

Shew that if one series of the lines of curvature is geodesic they are all repetitions of the same plane curve.

14. Shew that if the normal to a surface always passes through a given curve, one set of the lines of curvature are circles; and that those normals which pass through a given point on the curve are generating lines of a right cone whose axis is the tangent at that point. Hence shew that if the normal always passes through two curves, these curves must be conics in planes at right angles, the foci of one being the vertices of the other.

15. Find the differential equation of the projection on the plane xy of each family of lines of curvature of the surface which is the envelope of a sphere whose centre lies on the parabola $x^2 + 4ay = 0$, $z = 0$, and which passes through the origin.

16. Shew that the principal curvatures at any point of a surface are given by the equation

$$\begin{vmatrix} \dfrac{dl}{dx} + \dfrac{1}{\rho}, & \dfrac{dl}{dy}, & \dfrac{dl}{dz} \\ \dfrac{dm}{dx}, & \dfrac{dm}{dy} + \dfrac{1}{\rho}, & \dfrac{dm}{dz} \\ \dfrac{dn}{dx}, & \dfrac{dn}{dy}, & \dfrac{dn}{dz} + \dfrac{1}{\rho} \end{vmatrix} = 0,$$

where l, m, n are the direction-cosines of the normal at the point.

17. The tangent planes to the surface of centres at the two points where any normal meets it are at right angles.

18. Shew that the point for which $x = y = z$ is an umbilic of

$$x^m + y^m + z^m = a^m,$$

and the radius of curvature there is

$$\frac{a}{m-1}(3)^{\frac{m-2}{2m}}.$$

19. In a hyperbolic paraboloid, of which the principal parabolas are equal, the algebraic sum of the distances of all points of the same line of curvature from two fixed rectilinear generators is constant.

20. Along the normal at a point P of an ellipsoid is measured PQ of a length inversely proportional to the perpendicular from the centre on the tangent plane at P; prove that the locus of Q is another ellipsoid, and that the envelope of all such ellipsoids is the "surface of centres," that is the locus of the centres of principal curvature.

21. Shew that the specific curvature at any point of the surface $xyz = abc$ varies as the fourth power of the perpendicular from the origin on the tangent plane at the point, and that at an umbilicus it is $\frac{1}{3}(abc)^{-\frac{2}{3}}$.

22. If a surface have one principal radius of curvature constant it is the envelope of a sphere of constant radius.

23. Find the umbilici of the surface $\frac{x^3}{a} + \frac{y^3}{b} + \frac{z^3}{c} = k^2$, and shew that at the umbilicus $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$ the directions of the three lines of curvature are given by the equations

$$\frac{dx}{a} = \frac{dy}{b}, \quad \frac{dy}{b} = \frac{dz}{c} \text{ and } \frac{dz}{c} = \frac{dx}{a} \text{ respectively.}$$

24. If two geodesics be drawn on an ellipsoid from any point to two fixed points, the sine of the angle between them varies as the perpendicular on the tangent plane at the point.

25. Shew that on a surface of revolution, the distance of any point of a geodesic from the axis varies as the cosecant of the angle between the geodesic and the meridian.

26. If a geodesic line be drawn on a developable surface and cut any generating line of the surface at an angle ψ and at a distance t from the edge of regression measured along the generator, prove that

$$\frac{dt}{d\psi} + \cot\psi \,.\, t = \rho,$$

where ρ is the radius of curvature of the edge of regression at the point where the generator touches it.

27. Shew that the tangent to a geodesic or line of curvature on a quadric always touches a geodesic or line of curvature respectively on a confocal quadric.

28. Shew that the reciprocals of the radii of curvature and torsion of a curve drawn on a developable surface are

$$\frac{\sin^2\theta}{\rho\cos\alpha} \text{ and } \frac{\sin\theta\cos\theta}{\rho}+\frac{d\alpha}{ds},$$

where ρ is the principal radius of curvature of the surface at the point, θ the angle the tangent line to the curve makes with the generator through the point, and α the angle between the normal to the surface and the principal normal of the curve.

If a geodesic on a developable surface be a plane curve it must be one of the generators or else the surface must be a cylinder.

29. If $\frac{1}{\rho}$ and $\frac{1}{\sigma}$ be the curvature and tortuosity at any point of a geodesic drawn on a surface, and $\frac{1}{\rho_1}, \frac{1}{\rho_2}$ be the principal curvatures of the surface at that point, shew that

$$\frac{1}{\sigma^2}+\left(\frac{1}{\rho_1}-\frac{1}{\rho}\right)\left(\frac{1}{\rho_2}-\frac{1}{\rho}\right)=0.$$

30. Through a given generator of a hyperboloid of one sheet, draw a variable plane; this will touch the surface at some point A on the generator and will contain the normal to the surface at another point B. Shew that the sum of the square roots of the measures of curvature of the surface at A and B is constant for all planes through this generator.

Hence shew that the same proposition is true for any skew surface.

31. If ϖ be the pitch of the screw by which any generator of a skew surface twists into its consecutive position, shew that $\varpi^2+\rho\rho'=0$, where ρ, ρ' are the principal radii of curvature at the point where the shortest distance between the two consecutive generators meets them.

32. If a geodesic be drawn on an ellipsoid from an umbilicus to an extremity of the mean axis, prove that its radius of torsion at the latter point is

$$\frac{a^2c^2}{b\sqrt{a^2-b^2}\sqrt{b^2-c^2}},$$

where a, b, c are the semi-axes of the ellipsoid arranged in descending order of magnitude.

33. If from any point on a surface a number of geodesic lines be drawn in all directions, shew (1) that those which have the greatest and least torsion bisect the angles between the principal sections, and (2) that the radius of torsion of any line, making an angle θ with a principal section, is given by the equation

$$\frac{1}{R}=\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)\sin\theta\cos\theta,$$

where ρ_1, ρ_2 are the radii of curvature of the principal sections.

34. Find the equation to the surface which is the locus of the central circular sections of a series of confocal ellipsoids. Prove that this surface cuts all the ellipsoids orthogonally, and that the orthogonal trajectories of the circles, drawn upon the surface, are lines of curvature upon two hyperboloids confocal with the ellipsoids.

35. If a cone of revolution circumscribe an ellipsoid, prove that the plane of contact divides the ellipsoid into two portions whose total curvatures are $2\pi(1+\sin\alpha)$ and $2\pi(1-\sin\alpha)$, where 2α is the vertical angle of the cone.

36. If any cylinder circumscribes an ellipsoid it divides it into portions whose integral curvatures are equal.

37. The measure of curvature at any point of the surface

$$\frac{x^2+y^2}{a^2}-\frac{z^2}{c^2}=1 \text{ is } \frac{c^2}{(c^2+r^2)^2},$$

where r is the length of the generator through the point cut off by the plane $z=0$.

38. Prove that, if radii be drawn to a sphere parallel to the principal normals at every point of a closed curve of continuous

curvature, the locus of their extremities divides the surface of the sphere into two equal parts.

Hence shew that the total curvature of a geodesic triangle on any surface is equal to the excess of its angles over two right angles.

39. Define the radius of geodesic curvature of a curve drawn upon a surface, and shew that at any point it is equal to $R \cot \phi$, where R is the radius of curvature of the normal section containing the tangent to the given curve, and ϕ is the inclination of the osculating plane to that section.

40. If a surface roll on a second surface without rotation about the common normal, and the trace on one surface is a geodesic, the trace on the other surface is a geodesic.

Hence prove that Gauss's measure of curvature is constant for all areas enclosed by geodesics.

MISCELLANEOUS EXAMPLES.

1. THE inclinations to the horizon of two lines which are at right angles to one another are α, β, the lines being on a plane inclined to the horizon at an angle θ; shew that $\sin^2\theta = \sin^2\alpha + \sin^2\beta$.

2. Shew that the volume of the tetrahedron of which a pair of opposite edges is formed by lengths r, r' on the straight lines whose equations are

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} \text{ and } \frac{x-a'}{l'} = \frac{y-b'}{m'} = \frac{z-c'}{n'},$$

is

$$\tfrac{1}{6}rr' \begin{vmatrix} a-a', & b-b', & c-c' \\ l \quad , & m \quad , & n \\ l' \quad , & m' \quad , & n' \end{vmatrix}.$$

3. A parallelogram of paper is creased along its shorter diagonal, and the two halves are folded so as to make an angle θ with each other: find the distance between the extremities of the longer diagonal, and prove that it is equal to the shorter, if $\sin^2\dfrac{\theta}{2} = \cot\alpha\cot\beta$, where α and β are the angles the sides make with the shorter diagonal.

4. The ends of a straight line lie on two fixed planes which are at right angles to one another, and the straight line subtends a right angle at each of two given points: shew that the locus of its middle point is a plane.

5. The equations of three straight lines are $y-z=1$, $x=0$; $z-x=1$, $y=0$; and $x-y=1$, $z=0$; prove that the locus of all straight lines which intersect the three lines is

$$x^2+y^2+z^2-2yz-2zx-2xy=1.$$

6. Three fixed lines are cut by any other line in the points A, B, C, and D is the point on the line ABC such that $\{ABCD\}$ is harmonic: shew that the locus of D is a straight line.

7. A point moves so that its perpendicular distances from two given lines are in a constant ratio: shew that its locus is an hyperboloid whose circular sections are perpendicular to the given lines.

8. A straight line slides upon two fixed straight lines in such a way that the part intercepted subtends a right angle at a fixed point: shew that the line generates a conicoid.

9. A sphere touches the six edges of a tetrahedron: shew that the three lines joining pairs of opposite points of contact will meet in a point.

10. A straight line moves in such a manner that each of four fixed points on the line is always on a given plane; shew that any other fixed point on the line describes a plane ellipse.

11. Any three points P, Q, R, and the polar planes of those points with reference to any conicoid are taken. PQ_1, PR_1 are the perpendiculars from P on the polar planes of Q and R respectively; QR_2, QP_2 are the perpendiculars from Q on the polar planes of R and P respectively; and RP_3, RQ_3 are the perpendiculars from R on the polar planes of P and Q respectively. Shew that $PQ_1 . QR_2 . RP_3 = PR_1 . QP_2 . RQ_3$.

12. Shew that, if the equation

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$$

represent two planes, the planes which bisect the angles between them are given by the equation

$$\begin{vmatrix} x, & y, & z \\ ax + hy + gz, & hx + by + fz, & gx + fy + cz \\ \dfrac{1}{af - gh}, & \dfrac{1}{bg - hf}, & \dfrac{1}{ch - fg} \end{vmatrix} = 0.$$

13. Shew that, if the equation

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$$

represent two planes, the product of the perpendiculars on the planes from the point (x, y, z) is

$$\frac{ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy}{(a+b+c)^2 + 4(f^2 - bc)^2 + 4(g^2 - ca)^2 + 4(h^2 - ab)^2}.$$

14. If $U \equiv (abcdlmnpqr)(xyzw)^2 = 0$ is the equation of a cone, shew that the co-ordinates of the vertex satisfy the equations

$$\frac{\frac{\partial U}{\partial a}}{\frac{\partial \Delta}{\partial a}} = \frac{\frac{\partial U}{\partial b}}{\frac{\partial \Delta}{\partial b}} = \dots = \frac{\frac{\partial U}{\partial l}}{\frac{\partial \Delta}{\partial l}} = \dots,$$

where Δ is the discriminant.

15. Shew that, if the equation

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0,$$

represent a paraboloid of revolution, $c = b \pm a$. Shew also that if $c = b + a$, the equations of the axis of the paraboloid will be

$$cz + w = 0, \quad (cx + u)\sqrt{a} + (cy + v)\sqrt{b} = 0.$$

16. Shew that the three principal planes of the surface

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 1$$

are given by the equations

$$\begin{vmatrix} ax + hy + gz, & hx + by + fz, & gx + fy + cz \\ Ax + Hy + Gz, & Hx + By + Fz, & Gx + Fy + Cz \\ x, & y, & z \end{vmatrix} = 0,$$

where $A, B, C\dots$ are the minors of a, b, c in the determinant

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}.$$

17. If r be any semi-axis of the conicoid

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 1,$$

prove that the values of r will be given by

$$\frac{gh}{gh - af + \frac{f}{r^2}} + \frac{hf}{hf - bg + \frac{g}{r^2}} + \frac{fg}{fg - ch + \frac{h}{r^2}} = 1.$$

18. The ellipse $b^2x^2 + a^2y^2 - a^2b^2 = 0$, $z = 0$ is a plane section of a cone whose equation, referred to its principal axes, is

$$\beta\gamma x^2 + \gamma\alpha y^2 + \alpha\beta z^2 = 0.$$

Shew that the vertex of the cone is on the curve

$$\left\{\frac{x^2 + y^2 + z^2 - a^2 - b^2}{\alpha + \beta + \gamma}\right\}^6 = \left\{\frac{a^2b^2 - b^2x^2 - a^2y^2 - (a^2 + b^2) z^2}{\beta\gamma + \gamma\alpha + \alpha\beta}\right\}^3$$
$$= \left\{\frac{a^2b^2z^2}{\alpha\beta\gamma}\right\}^2.$$

19. Shew that the conicoid $ax^2 + by^2 + cz^2 + d = 0$ is its own polar reciprocal with respect to any one of the conicoids

$$\pm ax^2 \pm by^2 \pm cz^2 \pm d = 0.$$

20. Find the locus of the centre of the sphere which passes through two circular sections of a conicoid which are of opposite systems and whose planes are equidistant from the centre.

21. Prove that the foci of sections of an ellipsoid made by a series of parallel planes lie on an ellipse.

22. Shew that the perpendicular from the centre on the tangent plane at any point of $\dfrac{x^2 + y^2}{a^2} - \dfrac{z^2}{c^2} = 1$ is $\dfrac{ac}{\sqrt{c^2 + r^2}}$, where r is the length of a generator through the point cut off by the plane of xy.

23. The six lines AB', $B'C$, CA', $A'B$, BC', $C'A$ are six generators of the hyperboloid $ax^2 + by^2 + cz^2 = 1$, and AB', $B'C$, CA', are respectively parallel to $A'B$, BC', $C'A$; shew that, if the parallelopiped of which the six generators are edges be completed, the corners which are not on the hyperboloid will be on

$$ax^2 + by^2 + cz^2 + 3 = 0.$$

24. Shew that at any point the rate per unit of length of generator at which the normal to the hyperboloid $\dfrac{x^2 + y^2}{a^2} - \dfrac{z^2}{c^2} = 1$ twists round a generator as we move along it is $\dfrac{c}{c^2 + r^2}$, where r is the distance, measured along the generator, of the point from the plane of xy.

25. $ABCDQ$ is a twisted polygon all whose angles are right angles; AB, CD lying on fixed straight lines. Shew that if A, B, C, D be any points on their respective lines, the locus of P or Q is an hyperboloid of one sheet.

26. If l be the latus-rectum of a parabola, and l_1, l_2, l_3 the latera recta of its orthogonal projections upon a rectangular system of co-ordinate planes making angles α, β and γ respectively with the plane of the original parabola, then

$$\frac{2}{l^{\frac{2}{3}}} = \frac{\cos^{\frac{4}{3}}\alpha}{l_1^{\frac{2}{3}}} + \frac{\cos^{\frac{4}{3}}\beta}{l_2^{\frac{2}{3}}} + \frac{\cos^{\frac{4}{3}}\gamma}{l_3^{\frac{2}{3}}}.$$

27. If the six points on a conicoid, normals at which meet in a point, are joined in pairs by three lines, prove that whatever set of joining lines is taken the sum of the squares of the semi-diameters parallel to them is constant.

28. A conicoid whose centre is D touches the three planes YOZ, ZOX, XOY in A, B, C respectively: shew that the lines through A, B, C parallel respectively to OX, OY, OZ, and the line OD are four generators of an hyperboloid of one sheet.

29. Three perpendicular tangent planes are drawn, one to each of three confocal conicoids: shew that the normals at the points of contact of the planes, and the line joining their point of intersection to the centre of the conicoids are generators of an hyperboloid of one sheet.

30. If any line through a fixed point O meet any number of fixed planes in the points A, B, C......, and on the line a point X be taken such that $\frac{1}{OX} = \frac{1}{OA} + \frac{1}{OB} + \frac{1}{OC} + \dots$; shew that the locus of X will be a plane.

31. If any line through a fixed point O meet any given surface in the points A, B, C, D..., and X be taken such that $\frac{1}{OX} = \frac{1}{OA} + \frac{1}{OB} + \frac{1}{OC} + \frac{1}{OD} + \dots$; then will the locus of X be a plane.

32. Two straight lines drawn in fixed directions through any point O meet a given surface in the points A, B, C, D... and A', B', C', D'...; shew that $\dfrac{OA \,.\, OB \,.\, OC \,.\, OD \ldots}{OA' \,.\, OB' \,.\, OC' \,.\, OD' \ldots}$ is constant.

33. Prove that the pedal of a helix with regard to any point on its axis is a curve lying on a hyperboloid of one sheet; and that, if the pitch of the helix be $\frac{1}{4}\pi$, this curve will cut perpendicularly all the generators of one system of the hyperboloid.

34. A curve is drawn on a sphere of radius a cutting all the meridians at a constant angle; shew (i) that the foot of the perpendicular from the centre of the sphere upon the osculating plane is the centre of curvature; (2) that if ρ, σ be the radii of curvature and torsion $\sigma\rho^2 = a^2$.

35. Prove that the shortest distance of the tangents at two points PQ of any curve is ultimately equal to $\dfrac{PQ^3}{6\rho\sigma}$, where ρ and σ are the radii of curvature and torsion.

36. Tangent planes to a conicoid are drawn at points along a line of curvature: shew that the perpendiculars from the centre on their planes lie on a quadric cone, that the different cones so formed are confocal, and that the focal lines of the cones are perpendicular to the circular sections of the conicoid.

37. A curve is drawn making a constant angle α with the axis of a paraboloid of revolution: prove (i) that its projection on a plane perpendicular to the axis is the involute of a circle of radius $l \cot \alpha$, (ii) that its radii of curvature ρ and torsion σ are given by the equations $\rho^2 \sin^2\alpha = \sigma^2 \sin^2\alpha \cos^2\alpha = r^2 - l^2 \cot^2\alpha$, where r is the distance of the point from the axis, and l is the semi-latus rectum of the generating parabola.

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